

# A COMMON STRUCTURE IN PBW BASES OF THE NILPOTENT SUBALGEBRA OF $U_q(\mathfrak{g})$ AND QUANTIZED ALGEBRA OF FUNCTIONS

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*Dedicated to Professors Anatol N. Kirillov and Tetsuji Miwa who taught us the joy of doing mathematics*

## Abstract

For a finite-dimensional simple Lie algebra  $\mathfrak{g}$ , let  $U_q^+(\mathfrak{g})$  be the positive part of the quantized universal enveloping algebra, and  $A_q(\mathfrak{g})$  be the quantized algebra of functions. We show that the transition matrix of the PBW bases of  $U_q^+(\mathfrak{g})$  coincides with the intertwiner between the irreducible  $A_q(\mathfrak{g})$ -modules labeled by two different reduced expressions of the longest element of the Weyl group of  $\mathfrak{g}$ . This generalizes the earlier result by Sergeev on  $A_2$  related to the tetrahedron equation and endows a new representation theoretical interpretation with the recent solution to the 3D reflection equation for  $C_2$ . Our proof is based on a realization of  $U_q^+(\mathfrak{g})$  in a quotient ring of  $A_q(\mathfrak{g})$ .

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra and  $U_q(\mathfrak{g})$  be the Drinfeld-Jimbo quantized enveloping algebra.  $U_q(\mathfrak{g})$  has the subalgebra  $U_q^+(\mathfrak{g})$  generated by the Chevalley generators  $e_1, \dots, e_n$ , ( $n = \text{rank } \mathfrak{g}$ ) corresponding to the simple roots. Denote by  $W = \langle s_1, \dots, s_n \rangle$  the Weyl group of  $\mathfrak{g}$  generated by the simple reflections  $s_1, \dots, s_n$ . It is well known (see for example [15]) that for each reduced expression  $w_0 = s_{i_1} \cdots s_{i_l}$  of the longest element  $w_0 \in W$ , one can associate the Poincaré-Birkhoff-Witt (PBW) basis of  $U_q^+(\mathfrak{g})$  having the form

$$E_{\mathbf{i}}^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}, \quad A = (a_1, \dots, a_l) \in (\mathbb{Z}_{\geq 0})^l,$$

where  $e_{\beta_i}^{(a_i)}$ 's are the divided powers of the positive root vectors determined by the choice  $\mathbf{i} = (i_1, \dots, i_l)$ . See Section 2.2. Let  $\{E_{\mathbf{j}}^A \mid A \in (\mathbb{Z}_{\geq 0})^l\}$  with  $\mathbf{j} = (j_1, \dots, j_l)$  be another PBW basis associated with a yet different reduced expression  $w_0 = s_{j_1} \cdots s_{j_l}$ . Following Lusztig [14], one expands a basis in terms of another as

$$E_{\mathbf{i}}^A = \sum_{B \in (\mathbb{Z}_{\geq 0})^l} \gamma_B^A E_{\mathbf{j}}^B$$

and obtains the transition coefficient  $\gamma_B^A$  uniquely. We have suppressed its dependence on  $\mathbf{i}, \mathbf{j}$  in the notation. Many remarkable properties are known for  $\gamma_B^A$  including the fact  $\gamma_B^A \in \mathbb{Z}[q]$ . See [14, Prop.2.3] for example.

In this paper we show that the transition coefficients  $\gamma = (\gamma_B^A)$  coincide with the matrix elements of the intertwiner between the irreducible  $A_q(\mathfrak{g})$ -modules labeled by two different reduced expressions of the longest element of the Weyl group of  $\mathfrak{g}$ . Here  $A_q(\mathfrak{g})$  denotes the quantized algebra of functions associated with  $\mathfrak{g}$ . It is a Hopf subalgebra of the dual  $U_q(\mathfrak{g})^*$  which has been studied from a variety of aspects. See [6, 11, 17, 18, 21, 22, 23] for example. Let us briefly recall the most relevant result to the present paper due to Vaksman and Soibelman

[21, 22, 23]. To each reduced expression of a (not necessarily longest) element  $w = s_{i_1} \cdots s_{i_r} \in W$ , one can associate an irreducible representation  $\pi_{\mathbf{i}}$  labeled by  $\mathbf{i} = (i_1, \dots, i_r)$  having the form

$$\pi_{\mathbf{i}} = \pi_{i_1} \otimes \cdots \otimes \pi_{i_r} : A_q(\mathfrak{g}) \rightarrow \text{End}(\mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_r}}),$$

where each component  $\pi_i : A_q(\mathfrak{g}) \rightarrow \text{End}(\mathcal{F}_{q_i})$  is the *fundamental representation* of  $A_q(\mathfrak{g})$  on the  $q$ -oscillator Fock space  $\mathcal{F}_{q_i} = \bigoplus_{m \geq 0} \mathbb{C}(q)|m\rangle$ . See Section 4.1. The two irreducible representations  $\pi_{\mathbf{i}}$  and  $\pi_{\mathbf{j}}$  with  $\mathbf{j} = (j_1, \dots, j_r)$  are isomorphic if  $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r} \in W$  are reduced expressions (Theorem 7). Thus one has the intertwiner  $\Phi = \Phi_{\mathbf{i}, \mathbf{j}} : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_r}} \rightarrow \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_r}}$  characterized by

$$\pi_{\mathbf{j}}(g) \circ \Phi = \Phi \circ \pi_{\mathbf{i}}(g) \quad \forall g \in A_q(\mathfrak{g})$$

up to an overall constant. Writing the basis of the Fock space  $\mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_r}}$  as  $|A\rangle = |a_1\rangle \otimes \cdots \otimes |a_r\rangle$  with  $A = (a_1, \dots, a_r) \in (\mathbb{Z}_{\geq 0})^r$ , we define the matrix elements of  $\Phi = (\Phi_B^A)$  by  $\Phi|B\rangle = \sum_A \Phi_B^A |A\rangle$  and the normalization  $\Phi_{0, \dots, 0}^{0, \dots, 0} = 1$ . Our main result (Theorem 11) is concerned with the longest element case  $r = l$  and is stated for each pair  $(\mathbf{i}, \mathbf{j})$  as  $\gamma_B^A = \Phi_B^A$ , i.e.,

$$\gamma = \Phi. \quad (1)$$

For a convenience we also introduce the “checked” intertwiner  $\Phi^\vee = \Phi \circ \sigma$ , where  $\sigma(|a_1\rangle \otimes \cdots \otimes |a_l\rangle) = |a_l\rangle \otimes \cdots \otimes |a_1\rangle$  is the reversal of the components.

Our work is inspired by recent developments in 3 dimensional (3D) integrable systems related to rank 2 cases. Recall the Zamolodchikov tetrahedron equation [26] and the Isaev-Kulish 3D reflection equation [8]:

$$R_{356} R_{246} R_{145} R_{123} = R_{123} R_{145} R_{246} R_{356}, \quad (2)$$

$$R_{456} R_{489} K_{3579} R_{269} R_{258} K_{1678} K_{1234} = K_{1234} K_{1678} R_{258} R_{269} K_{3579} R_{489} R_{456}. \quad (3)$$

They are equalities among the linear operators acting on the tensor product of 6 and 9 vector spaces, respectively. The indices specify the components in the tensor product on which the operators  $R$  and  $K$  act nontrivially. They serve as 3D analogue of the Yang-Baxter and reflection equations postulating certain factorization conditions of straight strings which undergo the scattering  $R$  and the reflection  $K$  by a boundary plane.

For  $\mathfrak{g} = A_2$ , Kapranov and Voevodsky [10] showed that  $R = \Phi^\vee \in \text{End}(\mathcal{F}_q^{\otimes 3})$  provides a solution to the tetrahedron equation (2). Moreover it was discovered by Sergeev [19] that the solution of the tetrahedron equation  $R$  in [2] (given also in [10] with misprint) is related with the transition matrix as  $\gamma = R \circ \sigma$ . Thus the equality (1) for  $\mathfrak{g} = A_2$  is a corollary of their results. Apart from the  $A_2$  case, it has been shown more recently [12] that  $K = \Phi^\vee$  for  $\mathfrak{g} = C_2$  yields the first nontrivial solution to the 3D reflection equation (3). See also [13] for  $\mathfrak{g} = B_2$ . These results motivated us to investigate the general  $\mathfrak{g}$  case and have led to (1). It is our hope that it provides a useful insight into higher dimensional integrable systems from the representation theory of quantum groups.

The layout of the paper is as follows. In Section 2, we summarize the definitions of  $U_q(\mathfrak{g})$  and PBW bases. In Section 3, we recall the basic facts on  $A_q(\mathfrak{g})$  following Kashiwara [11]. A fundamental role is played by the Peter-Weyl type Theorem 1. The relation with the Reshetikhin-Takhtadzhyan-Faddeev realization by generators and relations [18] is explained and its concrete forms are quoted for  $A_n, C_n$  and  $G_2$  [20] which will be of use in later sections. Our new result here is the construction of a certain quotient ring  $A_q(\mathfrak{g})_S$  of  $A_q(\mathfrak{g})$ . The special elements  $\sigma_i \in A_q(\mathfrak{g})$  (Definition 3) and  $\xi_i \in A_q(\mathfrak{g})_S$  (36) will play a key role in our proof of (1). In Section 4, we briefly review the representation theory of  $A_q(\mathfrak{g})$  in [21, 22, 23] and sketch the intertwiners for the rank 2 cases. Section 5 is devoted to the proof of the main theorem  $\gamma = \Phi$ . It reduces to the rank 2 cases and is done without recourse to explicit formulae of  $\gamma$  or  $\Phi$ . Our method is to identify their characterizations under the correspondence  $e_i \mapsto \xi_i$ . In Section 6, as an additional result we show that this map actually extends to an algebra homomorphism  $U_q^+(\mathfrak{g}) \rightarrow A_q(\mathfrak{g})_S$  for general  $\mathfrak{g}$ .

2. QUANTIZED ENVELOPING ALGEBRA  $U_q(\mathfrak{g})$ 

**2.1. Definition.** In this paper  $\mathfrak{g}$  stands for a finite-dimensional simple Lie algebra. Its weight lattice, simple roots, simple coroots, fundamental weights are denoted by  $P, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, \{\varpi_i\}_{i \in I}$  where  $I$  is the index set of the Dynkin diagram of  $\mathfrak{g}$ . The Cartan matrix  $(a_{ij})_{i,j \in I}$  is given by  $a_{ij} = \langle h_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ .

The quantized enveloping algebra  $U_q(\mathfrak{g})$  is an associative algebra over  $\mathbb{Q}(q)$  generated by  $\{e_i, f_i, k_i^{\pm 1} \mid i \in I\}$  satisfying the relations:

$$\begin{aligned} k_i k_j &= k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \\ k_i e_j k_i^{-1} &= q_i^{\langle h_i, \alpha_j \rangle} e_j, \quad k_i f_j k_i^{-1} = q_i^{-\langle h_i, \alpha_j \rangle} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(r)} e_j e_i^{(1-a_{ij}-r)} &= \sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(r)} f_j f_i^{(1-a_{ij}-r)} = 0 \quad (i \neq j). \end{aligned} \quad (4)$$

Here we use the following notations:  $q_i = q^{(\alpha_i, \alpha_i)/2}$ ,  $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$ ,  $[n]_i! = \prod_{m=1}^n [m]_i$ ,  $e_i^{(n)} = e_i^n/[n]_i!$ ,  $f_i^{(n)} = f_i^n/[n]_i!$ . We normalize the simple roots so that  $q_i = q$  when  $\alpha_i$  is a short root.  $U_q(\mathfrak{g})$  is a Hopf algebra. As its comultiplication we adopt the following one.

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i.$$

**2.2. PBW basis.** Let  $W$  be the Weyl group of  $\mathfrak{g}$ . It is generated by simple reflections  $\{s_i \mid i \in I\}$  obeying the relations:  $s_i^2 = 1$ ,  $(s_i s_j)^{m_{ij}} = 1$  ( $i \neq j$ ) where  $m_{ij} = 2, 3, 4, 6$  for  $\langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle = 0, 1, 2, 3$ , respectively. Let  $w_0$  be the longest element of  $W$  and fix a reduced expression  $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$ . Then every positive root occurs exactly once in

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \beta_l = s_{i_1} s_{i_2} \cdots s_{i_{l-1}}(\alpha_{i_l}).$$

Correspondingly, define elements  $e_{\beta_r} \in U_q(\mathfrak{g})$  ( $r = 1, \dots, l$ ) by

$$e_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(e_{i_r}). \quad (5)$$

Here  $T_i$  is the action of the braid group on  $U_q(\mathfrak{g})$  introduced by Lusztig [15]. It is an algebra automorphism and is given on the generators  $\{e_j\}$  by

$$T_i(e_i) = -k_i f_i, \quad T_i(e_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r e_i^{(r)} e_j e_i^{(-a_{ij}-r)} \quad (i \neq j).$$

$U_q(\mathfrak{g})$  has a subalgebra generated by  $\{e_i \mid i \in I\}$ , denoted by  $U_q^+(\mathfrak{g})$ . It is known that  $e_{\beta_r} \in U_q^+(\mathfrak{g})$  holds for any  $r$ .  $U_q^+(\mathfrak{g})$  has the so-called Poincaré-Birkhoff-Witt (PBW) basis. It depends on the reduced expression  $s_{i_1} s_{i_2} \cdots s_{i_l}$  of  $w_0$ . Set  $\mathbf{i} = (i_1, i_2, \dots, i_l)$  and define for  $A = (a_1, a_2, \dots, a_l) \in (\mathbb{Z}_{\geq 0})^l$

$$E_{\mathbf{i}}^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}. \quad (6)$$

Then  $\{E_{\mathbf{i}}^A \mid A \in (\mathbb{Z}_{\geq 0})^l\}$  forms a basis of  $U_q^+(\mathfrak{g})$ . We hope that the notations  $e_{i_r}$  with  $i_r \in I$  and  $e_{\beta_r}$  with a positive root  $\beta_r$  can be distinguished properly from the context.

3. QUANTIZED ALGEBRA OF FUNCTIONS  $A_q(\mathfrak{g})$ 

**3.1. Definition.** Following [11] we give the definition of the quantized algebra of functions  $A_q(\mathfrak{g})$ . It is valid for any symmetrizable Kac-Moody algebra  $\mathfrak{g}$ . Let  $O_{\text{int}}(\mathfrak{g})$  be the category of integrable left  $U_q(\mathfrak{g})$ -modules  $M$  such that, for any  $u \in M$ , there exists  $l \geq 0$  satisfying  $e_{i_1} \cdots e_{i_l} u = 0$  for any  $i_1, \dots, i_l \in I$ . Then  $O_{\text{int}}(\mathfrak{g})$  is semisimple and any simple object is isomorphic to the irreducible module  $V(\lambda)$  with dominant integral highest weight  $\lambda$ . Similarly, we can consider the category  $O_{\text{int}}(\mathfrak{g}^{\text{opp}})$  of integrable right  $U_q(\mathfrak{g})$ -modules  $M^r$  such that, for any  $v \in M^r$ , there exists  $l \geq 0$  satisfying  $v f_{i_1} \cdots f_{i_l} = 0$  for any  $i_1, \dots, i_l \in I$ .  $O_{\text{int}}(\mathfrak{g}^{\text{opp}})$  is also semisimple and any simple object is isomorphic to the irreducible module  $V^r(\lambda)$  with dominant

integral highest weight  $\lambda$ . Let  $u_\lambda$  (resp.  $v_\lambda$ ) be a highest-weight vector of  $V(\lambda)$  (resp.  $V^r(\lambda)$ ). Then there exists a unique bilinear form  $(\cdot, \cdot)$

$$V^r(\lambda) \otimes V(\lambda) \rightarrow \mathbb{Q}(q)$$

satisfying

$$\begin{aligned} (v_\lambda, u_\lambda) &= 1 \quad \text{and} \\ (vP, u) &= (v, Pu) \quad \text{for } v \in V^r(\lambda), u \in V(\lambda), P \in U_q(\mathfrak{g}). \end{aligned}$$

Let  $U_q(\mathfrak{g})^*$  be  $\text{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$  and  $\langle \cdot, \cdot \rangle$  be the canonical pairing between  $U_q(\mathfrak{g})^*$  and  $U_q(\mathfrak{g})$ . The comultiplication  $\Delta$  of  $U_q(\mathfrak{g})$  induces a multiplication of  $U_q(\mathfrak{g})^*$  by

$$\langle \varphi \varphi', P \rangle = \langle \varphi \otimes \varphi', \Delta(P) \rangle \quad \text{for } P \in U_q(\mathfrak{g}), \quad (7)$$

thereby giving  $U_q(\mathfrak{g})^*$  the structure of  $\mathbb{Q}(q)$ -algebra. It also has a  $U_q(\mathfrak{g})$ -bimodule structure by

$$\langle x\varphi y, P \rangle = \langle \varphi, yPx \rangle \quad \text{for } x, y, P \in U_q(\mathfrak{g}). \quad (8)$$

We define the subalgebra  $A_q(\mathfrak{g})$  of  $U_q(\mathfrak{g})^*$  by

$$A_q(\mathfrak{g}) = \{ \varphi \in U_q(\mathfrak{g})^*; U_q(\mathfrak{g})\varphi \text{ belongs to } O_{\text{int}}(\mathfrak{g}) \text{ and } \varphi U_q(\mathfrak{g}) \text{ belongs to } O_{\text{int}}(\mathfrak{g}^{\text{opp}}) \},$$

and call it the quantized algebra of functions.

The following theorem is the  $q$ -analogue of the Peter-Weyl theorem. See *e.g.* [11] for a proof.

**Theorem 1.** *As a  $U_q(\mathfrak{g})$ -bimodule  $A_q(\mathfrak{g})$  is isomorphic to  $\bigoplus_\lambda V^r(\lambda) \otimes V(\lambda)$ , where  $\lambda$  runs over all dominant integral weights, by the homomorphisms*

$$\Psi_\lambda : V^r(\lambda) \otimes V(\lambda) \rightarrow A_q(\mathfrak{g})$$

given by

$$\langle \Psi_\lambda(v \otimes u), P \rangle = (v, Pu)$$

for  $v \in V^r(\lambda), u \in V(\lambda)$ , and  $P \in U_q(\mathfrak{g})$ .

Let us now assume that  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra and let  $\mathcal{R}$  be the universal  $R$  matrix. For its explicit formula see *e.g.* [5, p.273]. For our purpose it is enough to know that

$$\mathcal{R} \in q^{(\text{wt} \cdot, \text{wt} \cdot)} \bigoplus_{\beta \in Q^+} (U_q^+)_{\beta} \otimes (U_q^-)_{-\beta}, \quad (9)$$

where  $q^{(\text{wt} \cdot, \text{wt} \cdot)}$  is an operator acting on the tensor product  $u_\lambda \otimes u_\mu$  of weight vectors  $u_\lambda, u_\mu$  of weight  $\lambda, \mu$  by  $q^{(\text{wt} \cdot, \text{wt} \cdot)}(u_\lambda \otimes u_\mu) = q^{(\lambda, \mu)} u_\lambda \otimes u_\mu$ ,  $Q_+ = \bigoplus_i \mathbb{Z}_{\geq 0} \alpha_i$ , and  $(U_q^\pm)_{\pm\beta}$  is the subspace of  $U_q^\pm(\mathfrak{g})$  spanned by root vectors corresponding to  $\pm\beta$ .

Fix  $\lambda$ , let  $\{u_j^\lambda\}$  and  $\{v_i^\lambda\}$  be bases of  $V(\lambda)$  and  $V^r(\lambda)$  such that  $(v_i^\lambda, u_j^\lambda) = \delta_{ij}$ , and  $\varphi_{ij}^\lambda = \Psi_\lambda(v_i^\lambda \otimes u_j^\lambda)$ . Let  $R$  be the so-called constant  $R$  matrix for  $V(\lambda) \otimes V(\mu)$ . Denoting the homomorphism  $U_q(\mathfrak{g}) \rightarrow \text{End}(V(\lambda))$  by  $\pi_\lambda$ , it is given as

$$R \propto (\pi_\lambda \otimes \pi_\mu)(\sigma \mathcal{R}), \quad (10)$$

where  $\sigma$  stands for the exchange of the first and second components. The scalar multiple is determined appropriately depending on  $\mathfrak{g}$ . The reason we apply  $\sigma$  is that it agrees to the convention of [18].  $R$  satisfies

$$R\Delta(x) = \Delta'(x)R \quad \text{for any } x \in U_q(\mathfrak{g})$$

where  $\Delta' = \sigma \circ \Delta$ . Define matrix elements  $R_{ij,kl}$  by  $R(u_k^\lambda \otimes u_l^\mu) = \sum_{i,j} R_{ij,kl} u_i^\lambda \otimes u_j^\mu$ . Define the right action of  $R$  on  $V^r(\lambda) \otimes V^r(\mu)$  in such a way that  $((v_i^\lambda \otimes v_j^\mu)R, u_k^\lambda \otimes u_l^\mu) = (v_i^\lambda \otimes v_j^\mu, R(u_k^\lambda \otimes u_l^\mu))$

holds. Then we have  $(v_i^\lambda \otimes v_j^\mu)R = \sum_{k,l} R_{ij,kl} v_k^\lambda \otimes v_l^\mu$ . From

$$\begin{aligned}
\sum_{m,p} R_{ij,mp} \langle \varphi_{mk}^\lambda \varphi_{pl}^\mu, x \rangle &= \sum_{m,p} R_{ij,mp} \langle \varphi_{mk}^\lambda \otimes \varphi_{pl}^\mu, \Delta(x) \rangle \\
&= \sum_{m,p} R_{ij,mp} (v_m^\lambda \otimes v_p^\mu, \Delta(x)(u_k^\lambda \otimes u_l^\mu)) \\
&= ((v_i^\lambda \otimes v_j^\mu)R, \Delta(x)(u_k^\lambda \otimes u_l^\mu)) \\
&= (v_i^\lambda \otimes v_j^\mu, R\Delta(x)(u_k^\lambda \otimes u_l^\mu)) \\
&= \sum_{m,p} (v_i^\lambda \otimes v_j^\mu, \Delta'(x)(u_m^\lambda \otimes u_p^\mu)) R_{mp,kl} \\
&= \sum_{m,p} (v_j^\mu \otimes v_i^\lambda, \Delta(x)(u_p^\mu \otimes u_m^\lambda)) R_{mp,kl} \\
&= \sum_{m,p} \langle \varphi_{jp}^\mu \otimes \varphi_{im}^\lambda, \Delta(x) \rangle R_{mp,kl} = \sum_{m,p} \langle \varphi_{jp}^\mu \varphi_{im}^\lambda, x \rangle R_{mp,kl}
\end{aligned}$$

for any  $x \in U_q(\mathfrak{g})$ , we have

$$\sum_{m,p} R_{ij,mp} \varphi_{mk}^\lambda \varphi_{pl}^\mu = \sum_{m,p} \varphi_{jp}^\mu \varphi_{im}^\lambda R_{mp,kl}. \quad (11)$$

We call such a relation *RTT* relation.

**3.2. Right quotient ring  $A_q(\mathfrak{g})_{\mathcal{S}}$ .** For later use we construct a certain right quotient ring of  $A_q(\mathfrak{g})$  by a suitable multiplicatively closed subset  $\mathcal{S}$ . Our reference is [16, Chapter 2].

Let  $R$  be a noncommutative ring with 1 and  $\mathcal{S}$  a multiplicatively closed subset of  $R$ . The following condition is called the right Ore condition:

(Ore) For any  $r \in R, s \in \mathcal{S}$ ,  $r\mathcal{S} \cap sR \neq \emptyset$ .

Set

$$\text{ass } \mathcal{S} = \{r \in R \mid rs = 0 \text{ for some } s \in \mathcal{S}\}.$$

Then under the right Ore condition  $\text{ass } \mathcal{S}$  turns out a two-sided ideal. Let  $\bar{\phantom{x}} : R \rightarrow R/\text{ass } \mathcal{S}$  denote the canonical projection. Suppose

(reg)  $\bar{\mathcal{S}}$  consists of regular elements, namely, elements  $x$  such that both  $xr = 0$  and  $rx = 0$  imply  $r = 0$ .

Then a theorem in [16, Chapter 2] states

**Theorem 2.** *The right quotient ring  $R_{\mathcal{S}}$  exists, if and only if (Ore) and (reg) are satisfied.*

By passing to the images by  $\bar{\phantom{x}}$ , it suffices to consider the case when  $\text{ass } \mathcal{S} = 0$ , and then elements of  $R_{\mathcal{S}}$  are of the form  $r/s$ . For  $r_i/s_i \in R/\mathcal{S}$  ( $i = 1, 2$ ) the addition and multiplication formulae are given by

$$r_1/s_1 + r_2/s_2 = (r_1u + r_2u')/(s_1u), \quad (r_1/s_1)(r_2/s_2) = (r_1v')/(s_2v), \quad (12)$$

where  $u, u', v, v'$  are so chosen that  $s_1u = s_2u'$  ( $u \in \mathcal{S}, u' \in R$ ),  $r_2v = s_1v'$  ( $v \in \mathcal{S}, v' \in R$ ).

Let us return to our case where  $R = A_q(\mathfrak{g})$ .

**Definition 3.** For any  $i \in I$ , let  $u_{w_0\varpi_i}$  (resp.  $v_{\varpi_i}$ ) be a lowest (resp. highest) weight vector of  $V(\varpi_i)$  (resp.  $V^r(\varpi_i)$ ). Set

$$\sigma_i = \Psi_{\varpi_i}(v_{\varpi_i} \otimes u_{w_0\varpi_i}).$$

See Lemma 5 and the comment following it for the characterization of  $\sigma_i$ .

**Proposition 4.** *Let  $\varphi_{\lambda\mu}$  be an element of  $A_q(\mathfrak{g})$  such that  $k_i\varphi_{\lambda\mu} = q_i^{(h_i, \mu)}\varphi_{\lambda\mu}$ ,  $\varphi_{\lambda\mu}k_i = q_i^{(h_i, \lambda)}\varphi_{\lambda\mu}$  for any  $i \in I$ . Then the following commutation relation holds:*

$$q^{(\varpi_i, \lambda)}\sigma_i\varphi_{\lambda\mu} = q^{(w_0\varpi_i, \mu)}\varphi_{\lambda\mu}\sigma_i.$$

In particular,  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for any  $i, j$ .

*Proof.* Without loss of generality one can assume  $\varphi_{\lambda\mu} = \Psi_\nu(v_\lambda \otimes u_\mu)$  for some  $\nu, v_\lambda \in V^r(\nu), u_\mu \in V(\nu)$  such that  $k_i u_\mu = q_i^{\langle h_i, \mu \rangle} u_\mu, v_\lambda k_i = q_i^{\langle h_i, \lambda \rangle} v_\lambda$ . In view of (9), (10) we have

$$R(u_{w_0 \varpi_i} \otimes u_\mu) = q^{(w_0 \varpi_i, \mu)} u_{w_0 \varpi_i} \otimes u_\mu, \quad (v_{\varpi_i} \otimes v_\lambda) R = q^{(\varpi_i, \lambda)} v_{\varpi_i} \otimes v_\lambda.$$

Then (11) implies the commutation relation. The second relation follows from the first one, since  $(\varpi_i, \varpi_i) = (w_0 \varpi_i, w_0 \varpi_i)$ .  $\square$

Let  $n$  be the rank of  $\mathfrak{g}$  and define

$$\mathcal{S} = \{\sigma_1^{m_1} \cdots \sigma_n^{m_n} \mid m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}\},$$

which is obviously multiplicatively closed subset of  $A_q(\mathfrak{g})$ .

**Lemma 5.** *Let  $s$  be a nonzero element in  $\text{Im } \Psi_\lambda$  satisfying  $f_i s = s f_i = 0$  for any  $i \in I$ . Then  $s \in \mathbb{Q}(q)^\times \sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n}$  where  $\mathbb{Q}(q)^\times = \mathbb{Q}(q) \setminus \{0\}$  and  $\lambda_i = \langle h_i, \lambda \rangle$ .*

*Proof.* By (7), (8)  $f_i \sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n} = \sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n} f_i = 0$  for any  $i$  and  $\sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n}$  belongs to  $\text{Im } \Psi_\lambda$ . By Theorem 1 such an element is unique up to an element of  $\mathbb{Q}(q)^\times$ .  $\square$

In particular  $\sigma_i$  is characterized as the unique element (up to an overall constant) in  $\text{Im } \Psi_{\varpi_i}$  such that  $f_j \sigma_i = \sigma_i f_j = 0$  for all  $j \in I$ . We remark that Theorem 1 implies that if a nonzero element  $\varphi_{\lambda, \mu} \in A_q(\mathfrak{g})$  satisfies the assumption of Proposition 4 and  $f_j \varphi_{\lambda, \mu} = \varphi_{\lambda, \mu} f_j = 0$  for all  $j$ , then  $\lambda = w_0 \mu$  must hold.

**Theorem 6.** *The right quotient ring  $A_q(\mathfrak{g})_{\mathcal{S}}$  exists.*

*Proof.* In view of Theorem 2 it is enough to show that

- (1) if  $\varphi \neq 0$ , then  $\varphi s \neq 0$  for any  $s \in \mathcal{S}$ ,
- (1') if  $\varphi \neq 0$ , then  $s \varphi \neq 0$  for any  $s \in \mathcal{S}$ , and
- (2) the right Ore condition is satisfied,

since (1) implies  $\text{ass } \mathcal{S} = 0$ , then (1) and (1') imply  $\overline{\mathcal{S}} = \mathcal{S}$  consists of regular elements.

Let us prove (1). Let  $\varphi = \sum_j \varphi_j$  be the two-sided weight decomposition. If  $\varphi_j s \neq 0$  for some  $j$ ,  $\varphi s \neq 0$  since the weights of  $\varphi_j s$  are distinct. Hence we can reduce the claim when  $\varphi$  is a weight vector. Suppose  $\varphi = \sum_\mu \varphi_\mu, \varphi_\mu \in \text{Im } \Psi_\mu$  and let  $\lambda$  be a maximal weight, with respect to the standard ordering on weights, such that  $\varphi_\lambda \neq 0$ . Choose sequences  $i_1, \dots, i_k$  and  $j_1, \dots, j_l$  such that  $f_{i_k} \cdots f_{i_1} \varphi_\lambda f_{j_1} \cdots f_{j_l}$  turns out a left-lowest and right-highest weight vector. Then by Lemma 5 it coincides with  $c s'$  with some  $c \in \mathbb{Q}(q)^\times, s' \in \mathcal{S}$ . Then

$$f_{i_k} \cdots f_{i_1} (\varphi s) f_{j_1} \cdots f_{j_l} = c' s' s + \cdots$$

with another  $c' \in \mathbb{Q}(q)^\times$ . By the maximality of  $\lambda$  the remaining part  $+\cdots$  in the right hand side does not contain the terms with the same two-sided weight. Hence  $\cdots = 0$ . Therefore, the left hand side is not 0 and we conclude  $\varphi s \neq 0$ .

(1') is similar. For (2) we can reduce the claim when  $\varphi$  is a weight vector, and in this case the claim is clear from Proposition 4.  $\square$

**3.3. Realization by generators and relations.** We consider the fundamental representation  $V(\varpi_1)$  of  $U_q(\mathfrak{g})$  for  $\mathfrak{g} = A_{n-1}, C_n, G_2$ . Set  $N = \dim V(\varpi_1)$ . It is known [6, 18] that  $A_q(\mathfrak{g})$  for  $\mathfrak{g} = A_{n-1}, C_n, G_2$  is realized as an associative algebra with appropriate generators  $(t_{ij})_{1 \leq i, j \leq N}$  corresponding to  $V^r(\varpi_1) \otimes V(\varpi_1)$  satisfying *RTT* relations

$$\sum_{m,p} R_{ij,mp} t_{mk} t_{pl} = \sum_{m,p} t_{jp} t_{im} R_{mp,kl} \quad (13)$$

and additional ones depending on  $\mathfrak{g}$ . See below for each  $\mathfrak{g}$  under consideration. In all cases, there exists a comultiplication  $\Delta : A_q \rightarrow A_q \otimes A_q$  given by

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}. \quad (14)$$

3.3.1.  $A_{n-1}$  case. We present formulae for  $A_q(A_{n-1})$ . In this case  $N = n$ . Let  $u_1$  and  $v_1$  be the highest-weight vectors of  $V(\varpi_1)$  and  $V^r(\varpi_1)$  such that  $(v_1, u_1) = 1$  and set  $u_j = f_{j-1}f_{j-2}\cdots f_1u_1, v_j = v_1e_1e_2\cdots e_{j-1}$  for  $2 \leq j \leq n$ . Then the constant  $R$  matrix is given by

$$\sum_{i,j,k,l} R_{ij,kl} E_{ik} \otimes E_{jl} = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i > j} E_{ij} \otimes E_{ji}$$

where  $E_{ij}$  is the matrix unit. Define  $t_{ij} = \Psi_{\varpi_1}(v_i \otimes u_j)$ . Then the  $RTT$  relations among  $(t_{ij})_{1 \leq i,j \leq N}$  read explicitly as follows.

$$\begin{aligned} [t_{ik}, t_{jl}] &= \begin{cases} 0 & (i < j, k > l), \\ (q - q^{-1})t_{jk}t_{il} & (i < j, k < l), \end{cases} \\ t_{ik}t_{jk} &= qt_{jk}t_{ik} \quad (i < j), \quad t_{ki}t_{kj} = qt_{kj}t_{ki} \quad (i < j). \end{aligned}$$

In  $A_{n-1}$  case we need another condition that the quantum determinant is 1, i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma)} t_{1\sigma_1} \cdots t_{n\sigma_n} = 1,$$

where  $\mathfrak{S}_n = W(A_{n-1})$  is the symmetric group of degree  $n$  and  $\ell(\sigma)$  is the length of  $\sigma$ .

According to Definition 3, we have  $\sigma_1 = t_{13}$  and  $\sigma_2 = t_{12}t_{23} - qt_{22}t_{13}$ . As an exposition, we note that  $\sigma_i e_i$  in (39) is derived from

$$\begin{aligned} \langle \sigma_1 e_1, P \rangle &= \langle t_{13} e_1, P \rangle = \langle v_1 e_1, Pu_3 \rangle = \langle v_2, Pu_3 \rangle = \langle t_{23}, P \rangle, \\ \langle \sigma_2 e_2, P \rangle &= \langle (t_{12} \otimes t_{23} - qt_{22} \otimes t_{13}) \Delta(e_2), \Delta(P) \rangle = \langle t_{12} k_2 \otimes t_{23} e_2 - qt_{22} e_2 \otimes t_{13}, \Delta(P) \rangle \\ &= \langle t_{12} \otimes t_{33} - qt_{32} \otimes t_{13}, \Delta(P) \rangle = \langle t_{12} t_{33} - qt_{32} t_{13}, P \rangle \end{aligned}$$

for any  $P \in U_q(A_2)$ . See e.g., [17] for an extensive treatment.

3.3.2.  $C_n$  case. We present formulae for  $A_q(C_n)$ . In this case  $N = 2n$ . Let  $u_1$  be the highest-weight vector of  $V(\varpi_1)$  and define  $u_j$  for  $2 \leq j \leq 2n$  recursively by  $u_{j+1} = f_j u_j$  ( $j \leq n$ ),  $-f_{2n-j} u_j$  ( $j > n$ ). Let  $\{v_i\}$  be the dual basis to  $\{u_i\}$  in  $V^r(\varpi_1)$ , namely,  $\{v_i\}$  are determined by  $(v_i, u_j) = \delta_{ij}$ . Then the constant  $R$  matrix is given by

$$\begin{aligned} \sum_{i,j,k,l} R_{ij,kl} E_{ik} \otimes E_{jl} &= q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j, j'} E_{ii} \otimes E_{jj} + q^{-1} \sum_i E_{ii} \otimes E_{i'i'} \\ &\quad + (q - q^{-1}) \sum_{i > j} E_{ij} \otimes E_{ji} - (q - q^{-1}) \sum_{i > j} \epsilon_i \epsilon_j q^{\varrho_i - \varrho_j} E_{ij} \otimes E_{i'j'}, \\ i' &= 2n + 1 - i, \quad \epsilon_i = 1 \quad (1 \leq i \leq n), \quad \epsilon_i = -1 \quad (n < i \leq 2n), \\ (\varrho_1, \dots, \varrho_{2n}) &= (n - 1, n - 2, \dots, 1, 0, 0, -1, \dots, -n + 1). \end{aligned}$$

Define  $t_{ij} = \Psi_{\varpi_1}(v_i \otimes u_j)$ . The  $RTT$  relations are given by (13) with the above  $R_{ij,kl}$ . Additional relations are given by

$$\sum_{j,k,l} C_{jk} C_{lm} t_{ij} t_{lk} = \sum_{j,k,l} C_{ij} C_{kl} t_{kj} t_{lm} = -\delta_{im} \quad (C_{ij} = \delta_{i,j'} \epsilon_i q^{\varrho_j}).$$

3.3.3.  $G_2$  case. We have  $N = 7$  in this case. We adopt the basis  $\{u_i\}$  of  $V(\varpi_1)$  that has the representation matrices given as in [20, eq.(29)], and let  $\{v_i\}$  the dual basis in  $V^r(\varpi_1)$ . Define  $t_{ij} = \Psi_{\varpi_1}(v_i \otimes u_j)$ . Then  $A_q(G_2)$  is generated by  $(t_{ij})_{1 \leq i,j \leq 7}$  satisfying (i) and (ii) given below.

- (i)  $RTT$  relations (13) with the structure constants specified by  $R_{ij,kl} = R_{kl}^{ij}$  in [20, eq.(33)].
- (ii) Additional relations

$$g^{ij} = \sum_{k,l} t_{jl} t_{ik} g^{kl}, \quad \sum_k f_k^{ij} t_{km} = \sum_{k,l} t_{jl} t_{ik} f_m^{kl}, \quad (15)$$

where  $g^{ij}$  and  $f_k^{ij}$  are given by [20, eqs.(30),(31)].

The relations [20, eqs.(20),(22)] are equivalent to (15) if the  $RTT$  relations are imposed. See the explanation after [20, Def.7]. Note also that we use the opposite indices of the Dynkin diagram to [20].

#### 4. REPRESENTATIONS OF $A_q(\mathfrak{g})$

**4.1. General remarks.** Let us recall the results in [23, 22] on the representations of  $A_q(\mathfrak{g})$  necessary in this paper. Consider the simplest example  $A_q(A_1)$  generated by  $t_{11}, t_{12}, t_{21}, t_{22}$  with the relations

$$\begin{aligned} t_{11}t_{21} &= qt_{21}t_{11}, & t_{12}t_{22} &= qt_{22}t_{12}, & t_{11}t_{12} &= qt_{12}t_{11}, & t_{21}t_{22} &= qt_{22}t_{21}, \\ [t_{12}, t_{21}] &= 0, & [t_{11}, t_{22}] &= (q - q^{-1})t_{21}t_{12}, & t_{11}t_{22} - qt_{12}t_{21} &= 1. \end{aligned}$$

Let  $\text{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$  be the  $q$ -oscillator algebra, *i.e.*, an associative algebra with the relations

$$\mathbf{k}\mathbf{a}^+ = q\mathbf{a}^+\mathbf{k}, \quad \mathbf{k}\mathbf{a}^- = q^{-1}\mathbf{a}^-\mathbf{k}, \quad \mathbf{a}^-\mathbf{a}^+ = \mathbf{1} - q^2\mathbf{k}^2, \quad \mathbf{a}^+\mathbf{a}^- = \mathbf{1} - \mathbf{k}^2. \quad (16)$$

It has a representation on the Fock space  $\mathcal{F}_q = \bigoplus_{m \geq 0} \mathbb{C}(q)|m\rangle$ :

$$\mathbf{k}|m\rangle = q^m|m\rangle, \quad \mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle. \quad (17)$$

In what follows, the symbols  $\mathbf{k}, \mathbf{a}^+, \mathbf{a}^-$  shall also be regarded as the elements from  $\text{End}(\mathcal{F}_q)$ . It is easy to check that the following map  $\pi$  defines an irreducible representation of  $A_q(A_1)$  on  $\mathcal{F}_q$ :

$$\pi : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} \mu \mathbf{a}^- & \alpha \mathbf{k} \\ -q\alpha^{-1}\mathbf{k} & \mu^{-1}\mathbf{a}^+ \end{pmatrix}, \quad (18)$$

where  $\alpha, \mu$  are nonzero parameters.

**Theorem 7** ([22, 23]).

- (1) For each vertex  $i$  of the Dynkin diagram of  $\mathfrak{g}$ ,  $A_q(\mathfrak{g})$  has an irreducible representation  $\pi_i$  factoring through (18) via  $A_q(\mathfrak{g}) \twoheadrightarrow A_{q_i}(\mathfrak{sl}_{2,i})$ . ( $\mathfrak{sl}_{2,i}$  denotes the  $\mathfrak{sl}_2$ -subalgebra of  $\mathfrak{g}$  associated to  $i$ .)
- (2) Irreducible representations of  $A_q(\mathfrak{g})$  are in one to one correspondence with the elements of the Weyl group  $W$  of  $\mathfrak{g}$ .
- (3) Let  $w = s_{i_1} \cdots s_{i_l} \in W$  be an reduced expression in terms of the simple reflections. Then the irreducible representation corresponding to  $w$  is isomorphic to  $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$ .

Actually the assertions (2) and (3) hold up to the degrees of freedom of the parameters  $\alpha, \mu$  in (18). See [22] for the detail. We call  $\pi_i$  ( $i = 1, \dots, \text{rank } \mathfrak{g}$ ) the *fundamental representations*. For simplicity we denote  $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$  by  $\pi_{i_1, \dots, i_l}$ .

A crucial corollary of Theorem 7 is the following:

If  $s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l} \in W$  are reduced expressions, then  $\pi_{i_1, \dots, i_l} \simeq \pi_{j_1, \dots, j_l}$ .

In particular, there exists the isomorphism  $\Phi : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}} \rightarrow \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$  characterized (up to an overall constant) by

$$\pi_{j_1, \dots, j_l}(g) \circ \Phi = \Phi \circ \pi_{i_1, \dots, i_l}(g) \quad \forall g \in A_q(\mathfrak{g}).$$

Here  $\pi_{i_1, \dots, i_l}(g = t_{ij})$  for example means the tensor product representation  $\sum_{r_1, \dots, r_{l-1}} \pi_{i_1}(t_{ir_1}) \otimes \cdots \otimes \pi_{i_l}(t_{r_{l-1}, j})$  obtained by the  $(l-1)$ -fold application of the coproduct (14).

Elements of the Fock space  $|m_1\rangle \otimes \cdots \otimes |m_l\rangle \in \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$  will simply be denoted by  $|m_1, \dots, m_l\rangle$ . We will always normalize the intertwiner by the condition  $\Phi|0, 0, \dots, 0\rangle = |0, 0, \dots, 0\rangle$ . The exchange of the  $i$  th and the  $j$  th tensor components from the left will be denoted by  $P_{ij}$ . In the remainder of this section we concentrate on  $A_q(\mathfrak{g})$  of rank 2 cases  $\mathfrak{g} = A_2, C_2$  and  $G_2$ , and present the concrete forms of the fundamental representations, definition of the intertwiners with a few examples of their matrix elements.



**4.2.  $A_2$  case.** Let  $T = (t_{ij})_{1 \leq i, j \leq 3}$  be the  $3 \times 3$  matrix of the generators of  $A_q(A_2)$ . The fundamental representations  $\pi_i : A_q(A_2) \rightarrow \text{End}(\mathcal{F}_q)$  ( $i = 1, 2$ ) are given by

$$\pi_1(T) = \begin{pmatrix} \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} & 0 \\ -q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 \mathbf{a}^- & \alpha_2 \mathbf{k} \\ 0 & -q\alpha_2^{-1} \mathbf{k} & \mu_2^{-1} \mathbf{a}^+ \end{pmatrix}, \quad (19)$$

where  $\alpha_i, \mu_i$  are nonzero parameters.

The Weyl group  $W = \langle s_1, s_2 \rangle$  is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \quad s_1 s_2 s_1 = s_2 s_1 s_2.$$

Thus we have the isomorphism  $\pi_{121} \simeq \pi_{212}$ . Let  $\Phi$  be the corresponding intertwiner and denote by  $R$  the checked intertwiner  $\Phi^\vee$  explained after (1).

$$\begin{aligned} \pi_{121} \Phi &= \Phi \pi_{212}, \quad \pi_{121} R = R \pi'_{212}, \quad \pi'_{212} = P_{13} \pi_{212} P_{13}, \\ R &= \Phi P_{13} \in \text{End}(\mathcal{F}_q^{\otimes 3}). \end{aligned}$$

For example  $\pi'_{212}(t_{ij}) = \sum_{k,l} \pi_2(t_{l,j}) \otimes \pi_1(t_{k,l}) \otimes \pi_2(t_{ik})$ . Define the matrix elements of  $R$  and its parameter-free part  $\mathcal{R}$  by

$$R|i, j, k\rangle = \sum_{a,b,c} R_{ijk}^{abc} |a, b, c\rangle, \quad R_{ijk}^{abc} = \mu_1^{a-j+k} \mu_2^{b-a-k} \mathcal{R}_{ijk}^{abc}.$$

Then the following properties are valid for  $\mathcal{R} = (\mathcal{R}_{ijk}^{abc})$  [12].

$$\mathcal{R}_{ijk}^{abc} \in \mathbb{Z}[q], \quad \mathcal{R}_{ijk}^{abc} = 0 \text{ unless } (a+b, b+c) = (i+j, j+k), \quad (20)$$

$$\mathcal{R}^{-1} = \mathcal{R}, \quad \mathcal{R}_{ijk}^{abc} = \mathcal{R}_{kji}^{cba}, \quad \mathcal{R}_{ijk}^{abc} = \frac{(q^2)_i (q^2)_j (q^2)_k}{(q^2)_a (q^2)_b (q^2)_c} \mathcal{R}_{abc}^{ijk}, \quad (21)$$

$$\mathcal{R}_{ijk}^{abc}|_{q=0} = \delta_{i,b+(a-c)_+} \delta_{j,\min(a,c)} \delta_{k,b+(c-a)_+}. \quad (22)$$

Here  $(q^2)_a = \prod_{m=1}^a (1 - q^{2m})$  and  $(y)_+ = \max(0, y)$ . Due to (20),  $\mathcal{R}$  is the infinite direct sum of finite dimensional matrices. An explicit formula of  $\mathcal{R}_{ijk}^{abc}$  was obtained in [10] (unfortunately with misprint) and in [1, eq.(59)] (in a different context and gauge including square roots). The formula exactly matching the present convention is [12, eq.(2.20)]. The  $\mathcal{R}$  satisfies [10] the tetrahedron equation (2).

**Example 8.** The following is the list of all the nonzero  $\mathcal{R}_{314}^{abc}$ .

$$\begin{aligned} \mathcal{R}_{314}^{041} &= -q^2(1 - q^4)(1 - q^6)(1 - q^8), \\ \mathcal{R}_{314}^{132} &= (1 - q^6)(1 - q^8)(1 - q^4 - q^6 - q^8 - q^{10}), \\ \mathcal{R}_{314}^{223} &= q^2(1 + q^2)(1 + q^4)(1 - q^6)(1 - q^6 - q^{10}), \\ \mathcal{R}_{314}^{314} &= q^6(1 + q^2 + q^4 - q^8 - q^{10} - q^{12} - q^{14}), \\ \mathcal{R}_{314}^{405} &= q^{12}. \end{aligned}$$

Thus  $\mathcal{R}_{314}^{abc}|_{q=0} = \delta_{a,1} \delta_{b,3} \delta_{c,2}$  in agreement with (22).

**4.3.  $C_2$  case.** We have  $(q_1, q_2) = (q, q^2)$ . Let  $T = (t_{ij})_{1 \leq i, j \leq 4}$  be the  $4 \times 4$  matrix of the generators of  $A_q(C_2)$ . We use  $\text{Osc}_{q^2} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K} \rangle$  in addition to  $\text{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$  (16). The fundamental representations  $\pi_i : A_q(C_2) \rightarrow \text{End}(\mathcal{F}_{q_i})$  ( $i = 1, 2$ ) are given by

$$\pi_1(T) = \begin{pmatrix} \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} & 0 & 0 \\ -q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mu_1 \mathbf{a}^- & -\alpha_1 \mathbf{k} \\ 0 & 0 & q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu_2 \mathbf{A}^- & \alpha_2 \mathbf{K} & 0 \\ 0 & -q^2 \alpha_2^{-1} \mathbf{K} & \mu_2^{-1} \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (23)$$

where  $\alpha_i, \mu_i$  are nonzero parameters.

The Weyl group  $W = \langle s_1, s_2 \rangle$  is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \quad s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2.$$

Thus we have the isomorphism  $\pi_{2121} \simeq \pi_{1212}$ . Let  $\Phi$  be the corresponding intertwiner and denote by  $K$  the checked intertwiner  $\Phi^\vee$ .

$$\begin{aligned} \pi_{2121}\Phi &= \Phi \pi_{1212}, \quad \pi_{2121}K = K \pi'_{2121}, \quad \pi'_{2121} = P_{14}P_{23}\pi_{1212}P_{14}P_{23}, \\ K &= \Phi P_{14}P_{23} \in \text{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q). \end{aligned}$$

Define the matrix elements of  $K$  and its parameter-free part  $\mathcal{K}$  by

$$K|i, j, k, l\rangle = \sum_{a, b, c, d} K_{ijkl}^{abcd} |a, b, c, d\rangle, \quad K_{ijkl}^{abcd} = \mu_1^{2(c-k)} \mu_2^{b-j} \mathcal{K}_{ijkl}^{abcd}.$$

Then the following properties are valid for  $\mathcal{K} = (\mathcal{K}_{ijkl}^{abcd})$  [12].

$$\mathcal{K}_{ijkl}^{abcd} \in \mathbb{Z}[q], \quad \mathcal{K}_{ijkl}^{abcd} = 0 \text{ unless } (a+b+c, b+2c+d) = (i+j+k, j+2k+l), \quad (24)$$

$$\mathcal{K}^{-1} = \mathcal{K}, \quad \mathcal{K}_{ijkl}^{abcd} = \frac{(q^4)_i (q^2)_j (q^4)_k (q^2)_l}{(q^4)_a (q^2)_b (q^4)_c (q^2)_d} \mathcal{K}_{abcd}^{ijkl}, \quad (25)$$

$$\begin{aligned} \mathcal{K}_{ijkl}^{abcd}|_{q=0} &= \delta_{i,a'} \delta_{j,b'} \delta_{k,c'} \delta_{l,d'}, \quad a' = x+a+b-d, \quad b' = c+d-x - \min(a, c+x), \\ &\quad c' = \min(a, c+x), \quad d' = b+(c-a+x)_+, \quad x = (c-a+(d-b)_+)_+. \end{aligned} \quad (26)$$

Due to (24),  $\mathcal{K}$  is the infinite direct sum of finite dimensional matrices. An explicit formula of  $\mathcal{K}_{ijkl}^{abcd}$  is available in [12, eqs.(3.27),(3.28)]. This  $\mathcal{K}$  and  $\mathcal{R}$  in Section 4.2 satisfy [12] the 3D reflection equation (3).

**Example 9.** The following is the list of all the nonzero  $\mathcal{K}_{2110}^{abcd}$ .

$$\begin{aligned} \mathcal{K}_{2110}^{1300} &= q^8(1-q^8), \\ \mathcal{K}_{2110}^{2110} &= -q^4(1-q^8+q^{14}), \\ \mathcal{K}_{2110}^{2201} &= -q^6(1+q^2)(1-q^2+q^4-q^6-q^{10}), \\ \mathcal{K}_{2110}^{3011} &= 1-q^8+q^{14}, \\ \mathcal{K}_{2110}^{3102} &= -q^{10}(1-q+q^2)(1+q+q^2), \\ \mathcal{K}_{2110}^{4003} &= q^4. \end{aligned}$$

Thus  $\mathcal{K}_{2110}^{abcd}|_{q=0} = \delta_{a,3} \delta_{b,0} \delta_{c,1} \delta_{d,1}$  in agreement with (26).

**4.4.  $G_2$  case.** We have  $(q_1, q_2) = (q, q^3)$ . Let  $T = (t_{ij})_{1 \leq i, j \leq 7}$  be the  $7 \times 7$  matrix of the generators of  $A_q(G_2)$ . We use  $\text{Osc}_{q^3} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K} \rangle$  in addition to  $\text{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$  (16). The fundamental representations  $\pi_i : A_q(G_2) \rightarrow \text{End}(\mathcal{F}_{q_i})$  ( $i = 1, 2$ ) are given by

$$\begin{aligned} \pi_1(T) &= \begin{pmatrix} \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} & 0 & 0 & 0 & 0 & 0 \\ -q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mu_1 \mathbf{a}^-)^2 & [2]_1 \alpha_1 \mu_1 \mathbf{k} \mathbf{a}^- & (\alpha_1 \mathbf{k})^2 & 0 & 0 \\ 0 & 0 & -q\alpha_1^{-1} \mu_1 \mathbf{a}^- \mathbf{k} & \mathbf{a}^- \mathbf{a}^+ - \mathbf{k}^2 & \alpha_1 \mu_1^{-1} \mathbf{k} \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & (q\alpha_1^{-1} \mathbf{k})^2 & -[2]_1 (\alpha_1 \mu_1)^{-1} \mathbf{k} \mathbf{a}^+ & (\mu_1^{-1} \mathbf{a}^+)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} \\ 0 & 0 & 0 & 0 & 0 & -q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ \end{pmatrix}, \\ \pi_2(T) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 \mathbf{A}^- & \alpha_2 \mathbf{K} & 0 & 0 & 0 & 0 \\ 0 & -q^3 \alpha_2^{-1} \mathbf{K} & \mu_2^{-1} \mathbf{A}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 \mathbf{A}^- & \alpha_2 \mathbf{K} & 0 \\ 0 & 0 & 0 & 0 & -q^3 \alpha_2^{-1} \mathbf{K} & \mu_2^{-1} \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (27)$$

where  $\alpha_i, \mu_i$  are nonzero parameters and  $[2]_1 = q + q^{-1}$  as defined after (4).

The Weyl group  $W = \langle s_1, s_2 \rangle$  is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \quad s_2 s_1 s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2 s_1 s_2.$$

Thus we have the isomorphism  $\pi_{212121} \simeq \pi_{121212}$ . Let  $\Phi$  be the corresponding intertwiner and denote by  $F$  the checked intertwiner  $\Phi^\vee$ .

$$\pi_{212121} \Phi = \Phi \pi_{121212}, \quad \pi_{212121} F = F \pi'_{212121}, \quad \pi'_{212121} = P_{16} P_{25} P_{34} \pi_{121212} P_{16} P_{25} P_{34}, \quad (28)$$

$$F = \Phi P_{16} P_{25} P_{34} \in \text{End}(\mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q).$$

Define the matrix elements of  $F$  and its parameter-free part  $\mathcal{F}$  by

$$F|i, j, k, l, m, n\rangle = \sum_{a, b, c, d, e, f} F_{ijklmn}^{abcdef} |a, b, c, d, e, f\rangle, \\ F_{ijklmn}^{abcdef} = \mu_1^{3c-3k+d-l+3e-3m} \mu_2^{2k-2c+l-d+3m-3e+n-f} \mathcal{F}_{ijklmn}^{abcdef}.$$

Then the following properties are valid for  $\mathcal{F} = (\mathcal{F}_{ijklmn}^{abcdef})$ .

$$\mathcal{F}_{ijklmn}^{abcdef} \in \mathbb{Z}[q], \quad \mathcal{F}_{ijklmn}^{abcdef} = 0 \quad \text{unless} \quad \begin{pmatrix} a+b+2c+d+e \\ b+3c+2d+3e+f \end{pmatrix} = \begin{pmatrix} i+j+2k+l+m \\ j+3k+2l+3m+n \end{pmatrix}, \quad (29)$$

$$\mathcal{F}^{-1} = \mathcal{F}, \quad \mathcal{F}_{ijklmn}^{abcdef} = \frac{(q^6)_i (q^2)_j (q^6)_k (q^2)_l (q^6)_m (q^2)_n}{(q^6)_a (q^2)_b (q^6)_c (q^2)_d (q^6)_e (q^2)_f} \mathcal{F}_{ijklmn}^{abcdef}. \quad (30)$$

Due to (29),  $\mathcal{F}$  is the infinite direct sum of finite dimensional matrices. The formula for  $\mathcal{F}_{ijklmn}^{abcdef}|_{q=0}$  can be deduced by the ultradiscretization (tropical form) of [3, Th.3.1(c)]. Although a tedious algorithm can be formulated for calculating any given  $\mathcal{F}_{ijklmn}^{abcdef}$  by using (28), an explicit formula for it is yet to be constructed.

**Example 10.** The following is the list of all the nonzero  $\mathcal{F}_{010101}^{abcdef}$ .

$$\begin{aligned} \mathcal{F}_{010101}^{000200} &= q^4(1-q^2)(1-q^2-q^4-q^6), & \mathcal{F}_{010101}^{001001} &= -q(1-q^2)(1-q^2-q^4+q^8+q^{10}), \\ \mathcal{F}_{010101}^{010010} &= -q(1-q^2)(1-q^2-q^4+q^8+q^{10}), & \mathcal{F}_{010101}^{010101} &= 1-2q^2+2q^6+3q^8-2q^{12}-2q^{14}-q^{16}, \\ \mathcal{F}_{010101}^{020002} &= q^4(-2+2q^6+q^8+q^{10}), & \mathcal{F}_{010101}^{100011} &= -q^3(1-q^2)(1-q^6-q^8), \\ \mathcal{F}_{010101}^{100102} &= q(1-q^2-q^4-q^6+q^{10}+q^{12}+q^{14}), & \mathcal{F}_{010101}^{200004} &= q^4, \\ \mathcal{F}_{010101}^{110003} &= q(1-q+q^2)(1+q+q^2)(1-q^2-q^8). \end{aligned}$$

## 5. MAIN THEOREM

In this section we fix two reduced words  $\mathbf{i} = (i_1, \dots, i_l)$ ,  $\mathbf{j} = (j_1, \dots, j_l)$  of the longest element  $w_0 \in W$ .

**5.1. Definitions of  $\gamma_B^A$  and  $\Phi_B^A$ .** In the  $U_q(\mathfrak{g})$  side, we defined the PBW bases  $E_i^A, E_j^B$  of  $U_q^+(\mathfrak{g})$  in Section 2.2. We define their transition coefficient  $\gamma_B^A$  by

$$E_i^A = \sum_B \gamma_B^A E_j^B.$$

While, in the  $A_q(\mathfrak{g})$  side, we have the intertwiner  $\Phi : \mathcal{F}_{q_{i_1}} \otimes \dots \otimes \mathcal{F}_{q_{i_l}} \rightarrow \mathcal{F}_{q_{j_1}} \otimes \dots \otimes \mathcal{F}_{q_{j_l}}$  satisfying

$$\pi_j(g) \circ \Phi = \Phi \circ \pi_i(g) \quad \forall g \in A_q(\mathfrak{g}). \quad (31)$$

We take the parameters  $\mu, \alpha$  in (18) to be 1. This in particular means for rank 2 cases that  $\mu_i, \alpha_i$  entering  $\pi_i(T)$  in (19), (23) and (27) are all 1. The intertwiner  $\Phi$  is normalized by  $\Phi|0, 0, \dots, 0\rangle = |0, 0, \dots, 0\rangle$ . Under these conditions a matrix element  $\Phi_B^A$  of  $\Phi$  is uniquely specified by

$$\Phi|B\rangle = \sum_A \Phi_B^A |A\rangle,$$

where  $A = (a_1, \dots, a_l) \in (\mathbb{Z}_{\geq 0})^l$  and  $|A\rangle = |a_1\rangle \otimes \dots \otimes |a_l\rangle \in \mathcal{F}_{q_{j_1}} \otimes \dots \otimes \mathcal{F}_{q_{j_l}}$  and similarly for  $|B\rangle \in \mathcal{F}_{q_{i_1}} \otimes \dots \otimes \mathcal{F}_{q_{i_l}}$ . Then our main result is

**Theorem 11.**

$$\gamma_B^A = \Phi_B^A.$$

For any pair  $(\mathbf{i}, \mathbf{j})$ , from  $\mathbf{i}$  one can reach  $\mathbf{j}$  by applying Coxeter relations. In view of the uniqueness of  $\gamma$  and  $\Phi$  and the fact that the braid group action  $T_i$  is an algebra homomorphism, the proof of this theorem reduces to establishing the same equality for all  $\mathbf{g}$  of rank 2. This will be done in the rest of this section.

**5.2. Proof of Theorem 11 for rank 2 cases.** In the rank 2 cases, there are two reduced expressions  $s_{i_1} \dots s_{i_l}$  for the longest element of the Weyl group. Denote the associated sequences  $\mathbf{i} = (i_1, \dots, i_l)$  by  $\mathbf{1}, \mathbf{2}$  and set  $\mathbf{1}' = \mathbf{2}, \mathbf{2}' = \mathbf{1}$ . Concretely, we take them as

$$\begin{aligned} A_2 : \mathbf{1} &= (1, 2, 1), & \mathbf{2} &= (2, 1, 2), & (q_1, q_2) &= (q, q), \\ C_2 : \mathbf{1} &= (1, 2, 1, 2), & \mathbf{2} &= (2, 1, 2, 1), & (q_1, q_2) &= (q, q^2), \\ G_2 : \mathbf{1} &= (1, 2, 1, 2, 1, 2), & \mathbf{2} &= (2, 1, 2, 1, 2, 1), & (q_1, q_2) &= (q, q^3), \end{aligned}$$

where  $q_i$  defined after (4) is also recalled. In order to simplify the formulae in Section 5.3, we use the PBW bases and the Fock states in yet another normalization as follows:

$$\begin{aligned} \tilde{E}_i^A &:= ([a_1]_{i_1}! \dots [a_l]_{i_l}!) E_i^A = e_{\beta_1}^{a_1} \dots e_{\beta_l}^{a_l}, \\ |A\rangle &:= d_{i_1, a_1} \dots d_{i_l, a_l} |A\rangle, \quad d_{i, a} = q_i^{-a(a-1)/2} \lambda_i^a, \quad \lambda_i = (1 - q_i^2)^{-1}, \end{aligned} \quad (32)$$

where  $A = (a_1, \dots, a_l)$ . See after (4) for the symbol  $[a]_i!$ .  $e_{\beta_r}$  is defined in (5). Accordingly we introduce the matrix elements  $\tilde{\gamma}_B^A$  and  $\tilde{\Phi}_B^A$  by

$$\tilde{E}_i^A = \sum_B \tilde{\gamma}_B^A \tilde{E}_i^B, \quad \Phi |B\rangle = \sum_A \tilde{\Phi}_B^A |A\rangle, \quad (\mathbf{i} = \mathbf{1}, \mathbf{2}).$$

It follows that  $\gamma_B^A = \tilde{\gamma}_B^A \prod_{k=1}^l ([b_k]_{i_k}! / [a_k]_{i_k}!)$  and  $\Phi_B^A = \tilde{\Phi}_B^A \prod_{k=1}^l (d_{i_k, a_k} / d_{i_k, b_k})$  for  $B = (b_1, \dots, b_l)$ . On the other hand, we know  $\Phi_B^A = \Phi_A^B \prod_{k=1}^l ((q_{i_k}^2)_{b_k} / (q_{i_k}^2)_{a_k})$  from (21), (25) and (30). Due to the identity  $(q_i^2)_m d_{i, m} = [m]_i!$ , the assertion  $\gamma_B^A = \Phi_B^A$  of Theorem 11 is equivalent to

$$\tilde{\gamma}_B^A = \tilde{\Phi}_A^B. \quad (33)$$

Let  $\rho_i(x) = (\rho_i(x)_{AB})$  be the matrix for the left multiplication of  $x \in U_q^+(\mathfrak{g})$ :

$$x \cdot \tilde{E}_i^A = \sum_B \tilde{E}_i^B \rho_i(x)_{BA}. \quad (34)$$

Let further  $\pi_i(g) = (\pi_i(g)_{AB})$  be the representation matrix of  $g \in A_q(\mathfrak{g})$ :

$$\pi_i(g) |A\rangle = \sum_B |B\rangle \pi_i(g)_{BA}. \quad (35)$$

The following element in the right quotient ring  $A_q(\mathfrak{g})_{\mathcal{S}}$  will play a key role in our proof.

$$\xi_i = \lambda_i(\sigma_i e_i) / \sigma_i \quad (i = 1, 2). \quad (36)$$

See Definition 3 for  $\sigma_i$  and (39), (41), (42) for the concrete forms in rank 2 cases. In Section 5.3 we will check the following statement case by case.

**Proposition 12.** *For  $\mathbf{g}$  of rank 2,  $\pi_i(\sigma_i)$  is invertible and the following equality is valid:*

$$\rho_i(e_i)_{AB} = \pi_i(\xi_i)_{AB} \quad (i = 1, 2), \quad (37)$$

where the right hand side means  $\lambda_i \pi_i(\sigma_i e_i) \pi_i(\sigma_i)^{-1}$ .

*Proof of Theorem 11 for rank 2 case.* We write the both sides of (37) as  $M_{AB}^i$  and the one for  $\mathbf{i}'$  instead of  $\mathbf{i}$  as  $M_{AB}^{i'}$ . From

$$\sum_{B,C} \tilde{E}_{\mathbf{i}'}^C M_{CB}^i \tilde{\gamma}_B^A = e_i \sum_B \tilde{E}_{\mathbf{i}'}^B \tilde{\gamma}_B^A = e_i \tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{E}_{\mathbf{i}}^B M_{BA}^i = \sum_{B,C} \tilde{E}_{\mathbf{i}'}^C \tilde{\gamma}_C^B M_{BA}^i$$

we have  $\sum_B M_{CB}^i \tilde{\gamma}_B^A = \sum_B \tilde{\gamma}_C^B M_{BA}^i$ . On the other hand, the action of the two sides of (31) with  $g = \xi_i$  and  $\mathbf{j} = \mathbf{i}'$  are calculated as

$$\pi_{\mathbf{i}'}(\xi_i) \circ \Phi|A\rangle\rangle = \pi_{\mathbf{i}'}(\xi_i) \sum_B |B\rangle\rangle \tilde{\Phi}_A^B = \sum_{B,C} |C\rangle\rangle M_{CB}^i \tilde{\Phi}_A^B$$

and

$$\Phi \circ \pi_{\mathbf{i}}(\xi_i)|A\rangle\rangle = \Phi \sum_B |B\rangle\rangle M_{BA}^i = \sum_{B,C} |C\rangle\rangle \tilde{\Phi}_B^C M_{BA}^i.$$

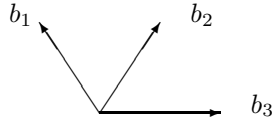
Hence  $\sum_B M_{CB}^i \tilde{\Phi}_A^B = \sum_B \tilde{\Phi}_B^C M_{BA}^i$ . Thus  $\tilde{\gamma}_B^A$  and  $\tilde{\Phi}_A^B$  satisfy the same relation. Moreover the maps  $\pi_{\mathbf{i}}$  and  $\rho_{\mathbf{i}}$  are both homomorphism, i.e.,  $\pi_{\mathbf{i}}(gh) = \pi_{\mathbf{i}}(g)\pi_{\mathbf{i}}(h)$  and  $\rho_{\mathbf{i}}(xy) = \rho_{\mathbf{i}}(x)\rho_{\mathbf{i}}(y)$ . We know that  $\Phi$  is the intertwiner of the irreducible  $A_q(\mathfrak{g})$  modules and (33) obviously holds as  $1 = 1$  at  $A = B = (0, \dots, 0)$ . Thus it is valid for arbitrary  $A$  and  $B$ .  $\square$

**Conjecture 13.** *The equality (37) is valid for any  $\mathfrak{g}$ .*

**5.3. Explicit formulae for rank 2 cases: Proof of Proposition 12.** Here we present the explicit formulae of (34) with  $x = e_i$  and (35) with  $g = \sigma_i, \sigma_i e_i$  that allow one to check Proposition 12. We use the notation  $\langle i \rangle = q^i - q^{-i}$ . In each case, there are two  $\mathbf{i}$ -sequences,  $\mathbf{1}$  and  $\mathbf{2} = \mathbf{1}'$  corresponding to the two reduced words. Let  $\chi$  be the anti-algebra involution such that  $\chi(e_i) = e_i$ . Then the relation  $\chi(\tilde{E}_{\mathbf{i}'}^A) = \tilde{E}_{\mathbf{i}'}^{\bar{A}}$  holds, where  $\bar{A} = (a_l, \dots, a_2, a_1)$  denotes the reversal of  $A = (a_1, a_2, \dots, a_l)$ . Applying  $\chi$  to (34) with  $x = e_i$  yields the right multiplication formula  $\tilde{E}_{\mathbf{i}'}^{\bar{A}} \cdot e_i = \sum_B \tilde{E}_{\mathbf{i}'}^B \rho_{\mathbf{i}}(e_i)_{BA}$  for  $\mathbf{i}'$ -sequence. In view of this fact, we shall present the left and right multiplication formulae for  $\mathbf{i} = \mathbf{2}$  only.

As for (35) with  $g = \xi_i$  in (36), explicit formulae for  $\sigma_i, \sigma_i e_i \in A_q(\mathfrak{g})$  and their image by the both representations  $\pi_1$  and  $\pi_2$  will be given. We include an exposition on how to use these data to check (37) along the simplest  $A_2$  case. The  $C_2$  and  $G_2$  cases are similar.

**5.3.1.  $A_2$  case.**



The  $q$ -Serre relations are

$$e_1^2 e_2 - [2]_1 e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - [2]_1 e_2 e_1 e_2 + e_1 e_2^2 = 0,$$

where  $[m]_1 = \langle m \rangle / \langle 1 \rangle$ . Let  $b_1, b_2, b_3$  be the generator for positive roots:  $b_1 = e_2$ ,  $b_2 = e_1 e_2 - q e_2 e_1$  and  $b_3 = e_1$ . In the notation of Section 2.2, they are the root vectors  $b_i = e_{\beta_i}$  associated with the reduced expression  $w_0 = s_2 s_1 s_2$  for  $\mathbf{2} = (2, 1, 2)$ . The corresponding positive roots are  $(\beta_1, \beta_2, \beta_3) = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1)$ . In particular,  $b_2 = T_2(e_1)$ . Their commutation relations are  $b_2 b_1 = q^{-1} b_1 b_2$ ,  $b_3 b_1 = b_2 + q b_1 b_3$ ,  $b_3 b_2 = q^{-1} b_2 b_3$ .

**Lemma 14.** *For  $\tilde{E}_2^{a,b,c} = b_1^a b_2^b b_3^c$ , we have*

$$\begin{aligned} \tilde{E}_2^{a,b,c} \cdot e_1 &= \tilde{E}_2^{a,b,c+1}, \\ \tilde{E}_2^{a,b,c} \cdot e_2 &= q^{c-b} \tilde{E}_2^{a+1,b,c} + [c]_1 \tilde{E}_2^{a,b+1,c-1}, \\ e_1 \cdot \tilde{E}_2^{a,b,c} &= q^{a-b} \tilde{E}_2^{a,b,c+1} + [a]_1 \tilde{E}_2^{a-1,b+1,c}, \end{aligned}$$

$$e_2 \cdot \tilde{E}_2^{a,b,c} = \tilde{E}_2^{a+1,b,c}.$$

*Proof.* By induction, we have

$$\begin{aligned} b_3 b_1^n &= q^n b_1^n b_3 + [n]_1 b_1^{n-1} b_2, & b_3 b_2^n &= q^{-n} b_2^n b_3, \\ b_3^n b_1 &= q^n b_1 b_3^n + [n]_1 b_2 b_3^{n-1}, & b_2^n b_1 &= q^{-n} b_1 b_2^n. \end{aligned}$$

The lemma is a direct consequence of these formulae.  $\square$

Set  $\tilde{E}_1^{a,b,c} = \chi(\tilde{E}_2^{c,b,a}) = \chi(b_3^a) \chi(b_2^b) \chi(b_1^c) = b_3^a b_2^b b_1^c$ , where  $b'_2 := \chi(b_2) = e_2 e_1 - q e_1 e_2$ . By applying  $\chi$  to the first two relations in Lemma 14, we get

$$e_1 \cdot \tilde{E}_1^{a,b,c} = \tilde{E}_1^{a+1,b,c}, \quad e_2 \cdot \tilde{E}_1^{a,b,c} = q^{a-b} \tilde{E}_1^{a,b,c+1} + [a]_1 \tilde{E}_1^{a-1,b+1,c}. \quad (38)$$

Thus we find  $\rho_{i'}(e_i) = \rho_i(e_{3-i})$ . This property is only valid for  $A_2$  and not in  $C_2$  and  $G_2$ .

Let us turn to the representations  $\pi_i$  of  $A_q(A_2)$ . The elements  $\sigma_i$  in Definition 3 and  $\sigma_i e_i$  are given by

$$\sigma_1 = t_{13}, \quad \sigma_2 = t_{12} t_{23} - q t_{22} t_{13}, \quad \sigma_1 e_1 = t_{23}, \quad \sigma_2 e_2 = t_{12} t_{33} - q t_{32} t_{13}. \quad (39)$$

See the exposition at the end of Section 3.3.1 and the remark after Lemma 5.

From (14) and (19) with  $\alpha_i = \mu_i = 1$ , we find

$$\pi_1(\sigma_1) = \mathbf{k}_1 \mathbf{k}_2, \quad \pi_1(\sigma_1 e_1) = \mathbf{a}_1^+ \mathbf{k}_2, \quad \pi_1(\sigma_2) = \mathbf{k}_2 \mathbf{k}_3, \quad \pi_1(\sigma_2 e_2) = \mathbf{a}_1^- \mathbf{a}_2^+ \mathbf{k}_3 + \mathbf{k}_1 \mathbf{a}_3^+,$$

where the notation like  $\mathbf{k}_1 \mathbf{a}_3^+ = \mathbf{k} \otimes 1 \otimes \mathbf{a}^+$  has been used. Since  $\mathbf{k} \in \text{End}(\mathcal{F}_q)$  is invertible, so is  $\pi_i(\sigma_i)$  and we may write

$$\pi_1(\xi_1) = \lambda_1 \mathbf{a}_1^+ \mathbf{k}_1^{-1}, \quad \pi_1(\xi_2) = \lambda_2 (\mathbf{a}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + \mathbf{k}_1 \mathbf{k}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1}),$$

where  $\lambda_1 = \lambda_2 = (1 - q^2)^{-1}$ . The action of each component on the ket vector  $|m\rangle := d_{i,m}|m\rangle \in \mathcal{F}_{q_i}$  (cf. (32)) takes the form

$$\mathbf{a}^+ |m\rangle = \lambda_i^{-1} q_i^m |m+1\rangle, \quad \mathbf{a}^- |m\rangle = [m]_i |m-1\rangle, \quad \mathbf{k} |m\rangle = q_i^m |m\rangle, \quad (40)$$

due to (17). (The formula (40) is valid also for  $C_2$  and  $G_2$  provided that  $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$  are interpreted as  $\mathbf{A}^+, \mathbf{A}^-, \mathbf{K}$  for  $i = 2$ .) Thus one has

$$\pi_1(\xi_1) |a, b, c\rangle = |a+1, b, c\rangle, \quad \pi_1(\xi_2) |a, b, c\rangle = [a]_1 |a-1, b+1, c\rangle + q^{a-b} |a, b, c+1\rangle.$$

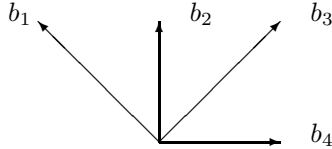
This agrees with (38) thereby proving (37) for  $\mathbf{i} = \mathbf{1}$ . The other case  $\mathbf{i} = \mathbf{2}$  also holds due to the symmetry  $\pi_2(\xi_i) = \pi_1(\xi_{3-i})$ . Thus Proposition 12 is established for  $A_2$ .

In terms of the checked intertwiner  $\mathcal{R}$  in Section 4.2, Theorem 11 implies

$$E_i^{a,b,c} = \sum_{i,j,k} \mathcal{R}_{ijk}^{abc} E_{i'}^{k,j,i}.$$

This is valid either for  $\mathbf{i} = \mathbf{1}$  or  $\mathbf{2}$  thanks to the middle property in (21). This relation connecting the PBW bases with the solution of the tetrahedron equation is due to [19].

5.3.2.  $C_2$  case.



The  $q$ -Serre relations are

$$\begin{aligned} e_1^3 e_2 - [3]_1 e_1^2 e_2 e_1 + [3]_1 e_1 e_2 e_1^2 - e_2 e_1^3 &= 0, \\ e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 &= 0, \end{aligned}$$

where  $[m]_1 = \langle m \rangle / \langle 1 \rangle$  and  $[m]_2 = \langle 2m \rangle / \langle 2 \rangle$ .

Let  $b_1, \dots, b_4$  be the generator for positive roots:  $b_1 = e_2$ ,  $b_2 = e_1 e_2 - q^2 e_2 e_1$ ,  $b_3 = \frac{1}{[2]_1}(e_1 b_2 - b_2 e_1)$  and  $b_4 = e_1$ . Their commutation relations are  $b_2 b_1 = q^{-2} b_1 b_2$ ,  $b_3 b_1 = -q^{-1} \langle 1 \rangle [2]_1^{-1} b_2^2 + b_1 b_3$ ,  $b_4 b_1 = b_2 + q^2 b_1 b_4$ ,  $b_3 b_2 = q^{-2} b_2 b_3$ ,  $b_4 b_2 = [2]_1 b_3 + b_2 b_4$ ,  $b_4 b_3 = q^{-2} b_3 b_4$ .

**Lemma 15.** For  $\tilde{E}_2^{a,b,c,d} = b_1^a b_2^b b_3^c b_4^d$ , we have

$$\begin{aligned} \tilde{E}_2^{a,b,c,d} \cdot e_1 &= \tilde{E}_2^{a,b,c,d+1}, \\ \tilde{E}_2^{a,b,c,d} \cdot e_2 &= [d]_1 q^{d-2c-1} \tilde{E}_2^{a,b+1,c,d-1} + q^{2(d-b)} \tilde{E}_2^{a+1,b,c,d} \\ &\quad - \langle 1 \rangle q^{2d-2c+1} [c]_2 [2]_1^{-1} \tilde{E}_2^{a,b+2,c-1,d} + [d-1]_1 [d]_1 \tilde{E}_2^{a,b,c+1,d-2}, \\ e_1 \cdot \tilde{E}_2^{a,b,c,d} &= [2]_1 [b]_1 q^{2a-b+1} \tilde{E}_2^{a,b-1,c+1,d} + q^{2a-2c} \tilde{E}_2^{a,b,c,d+1} + [a]_2 \tilde{E}_2^{a-1,b+1,c,d}, \\ e_2 \cdot \tilde{E}_2^{a,b,c,d} &= \tilde{E}_2^{a+1,b,c,d}. \end{aligned}$$

*Proof.* By induction, we have

$$\begin{aligned} b_4 b_1^n &= b_1^n b_4 q^{2n} + [n]_2 b_1^{n-1} b_2, \\ b_4 b_2^n &= [2]_1 [n]_1 b_2^{n-1} b_3 q^{-n+1} + b_2^n b_4, \\ b_4 b_3^n &= q^{-2n} b_3^n b_4, \\ b_4^n b_1 &= [n]_1 b_2 b_4^{n-1} q^{n-1} + b_1 b_4^n q^{2n} + [n-1]_1 [n]_1 b_3 b_4^{n-2}, \\ b_3^n b_1 &= -q^{1-2n} \langle 1 \rangle [n]_2 [2]_1^{-1} b_2^2 b_3^{n-1} + b_1 b_3^n, \\ b_3^n b_2 &= q^{-2n} b_2 b_3^n, \\ b_2^n b_1 &= q^{-2n} b_1 b_2^n. \end{aligned}$$

The lemma is a direct consequence of these formulae.  $\square$

Set  $\tilde{E}_1^{a,b,c,d} = \chi(\tilde{E}_2^{d,c,b,a})$ . The left multiplication formula for this basis is deduced from the above lemma by applying  $\chi$ . One can adjust the definition of  $E_i^A$  in (6) with that in [25] by setting  $v = q^{-1}$ .

Let us turn to the representations  $\pi_i$  of  $A_q(C_2)$ . The elements  $\sigma_i$  in Definition 3 and  $\sigma_i e_i$  are given by

$$\sigma_1 = t_{14}, \quad \sigma_2 = t_{13} t_{24} - q t_{23} t_{14}, \quad \sigma_1 e_1 = t_{24}, \quad \sigma_2 e_2 = t_{13} t_{34} - q t_{33} t_{14}. \quad (41)$$

From (14) and (23) with  $\alpha_i = \mu_i = 1$ , we have

$$\begin{aligned} \pi_1(\sigma_1) &= -\mathbf{k}_1 \mathbf{K}_2 \mathbf{k}_3, \\ \pi_1(\sigma_1 e_1) &= -\mathbf{a}_1^+ \mathbf{K}_2 \mathbf{k}_3, \\ \pi_1(\sigma_2) &= -\mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4, \\ \pi_1(\sigma_1 e_2) &= -\mathbf{a}_1^{-2} \mathbf{A}_2^+ \mathbf{k}_3^2 \mathbf{K}_4 - [2]_1 \mathbf{a}_1^- \mathbf{k}_1 \mathbf{a}_3^+ \mathbf{k}_3 \mathbf{K}_4 - \mathbf{k}_1^2 \mathbf{A}_2^- \mathbf{a}_3^{+2} \mathbf{K}_4 - \mathbf{A}_4^+ \mathbf{k}_1^2 \mathbf{K}_2, \\ \lambda_1^{-1} \pi_1(\xi_1) &= \mathbf{a}_1^+ \mathbf{k}_1^{-1}, \\ \lambda_2^{-1} \pi_1(\xi_2) &= \mathbf{a}_1^{-2} \mathbf{A}_2^+ \mathbf{K}_2^{-1} + \mathbf{k}_1^2 \mathbf{A}_2^- \mathbf{K}_2^{-1} \mathbf{a}_3^{+2} \mathbf{k}_3^{-2} + [2]_1 \mathbf{a}_1^- \mathbf{k}_1 \mathbf{K}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1} + \mathbf{k}_1^2 \mathbf{k}_3^{-2} \mathbf{A}_4^+ \mathbf{K}_4^{-1}, \\ \pi_2(\sigma_1) &= -\mathbf{k}_2 \mathbf{K}_3 \mathbf{k}_4, \\ \pi_2(\sigma_1 e_1) &= -\mathbf{K}_1 \mathbf{k}_2 \mathbf{a}_4^+ - \mathbf{K}_1 \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{k}_4 - \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4, \\ \pi_2(\sigma_2) &= -\mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3, \\ \pi_2(\sigma_2 e_2) &= -\mathbf{A}_1^+ \mathbf{k}_2^2 \mathbf{K}_3, \\ \lambda_1^{-1} \pi_2(\xi_1) &= \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + \mathbf{K}_1 \mathbf{a}_2^- \mathbf{k}_2^{-1} \mathbf{A}_3^+ \mathbf{K}_3^{-1} + \mathbf{K}_1 \mathbf{K}_3^{-1} \mathbf{a}_4^+ \mathbf{k}_4^{-1}, \\ \lambda_2^{-1} \pi_2(\xi_2) &= \mathbf{A}_1^+ \mathbf{K}_1^{-1}. \end{aligned}$$

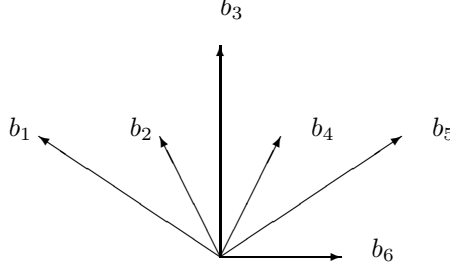
We find that  $\pi_i(\sigma_i)$  is invertible. Comparing these formulae with Lemma 15 by using (40), the equality (37) is directly checked. Thus Proposition 12 is established for  $C_2$ .

In terms of the checked intertwiner  $\mathcal{K}$  in Section 4.3, Theorem 11 implies

$$E_2^{a,b,c,d} = \sum_{i,j,k,l} \mathcal{K}_{ijkl}^{abcd} E_1^{l,k,j,i}.$$

Thus the solution to the 3D reflection equation [12] is identified with the transition coefficient of the PBW bases for  $U_q^+(C_2)$ .

#### 5.4. $G_2$ case.



The  $q$ -Serre relations are

$$\begin{aligned} e_1^4 e_2 - [4]_1 e_1^3 e_2 e_1 + [4]_1 [3]_1 [2]_1^{-1} e_1^2 e_2 e_1^2 - [4]_1 e_1 e_2 e_1^3 + e_2 e_1^4 &= 0, \\ e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 &= 0, \end{aligned}$$

where we remind that  $[m]_1 = \langle m \rangle / \langle 1 \rangle$  and  $[m]_2 = \langle 3m \rangle / \langle 3 \rangle$ .

Let  $b_1, \dots, b_6$  be the generator for positive roots:  $b_1 = e_2$ ,  $b_2 = e_1 e_2 - q^3 e_2 e_1$ ,  $b_4 = \frac{1}{[2]_1}(e_1 b_2 - q b_2 e_1)$ ,  $b_5 = \frac{q^2}{[3]_1}(e_1 b_4 - q^{-1} b_4 e_1)$ ,  $b_3 = \frac{1}{[3]_1}(b_4 b_2 - q^{-1} b_2 b_4)$  and  $b_6 = e_1$ . Their commutation relations are as follows:  $b_2 b_1 = b_1 b_2 q^{-3}$ ,  $b_3 b_1 = \langle 1 \rangle^2 b_3^3 q^{-3} [3]_1^{-1} + b_1 b_3 q^{-3}$ ,  $b_4 b_1 = b_1 b_4 - b_2^2 \langle 1 \rangle q^{-1}$ ,  $b_5 b_1 = b_1 b_5 q^3 - b_2 b_4 \langle 1 \rangle q^{-1} - (q^4 + q^2 - 1) b_3 q^{-3}$ ,  $b_6 b_1 = b_1 b_6 q^3 + b_2$ ,  $b_3 b_2 = b_2 b_3 q^{-3}$ ,  $b_4 b_2 = b_2 b_4 q^{-1} + b_3 [3]_1$ ,  $b_5 b_2 = b_2 b_5 - b_4^2 \langle 1 \rangle q^{-1}$ ,  $b_6 b_2 = q b_2 b_6 + b_4 [2]_1$ ,  $b_4 b_3 = b_3 b_4 q^{-3}$ ,  $b_5 b_3 = \langle 1 \rangle^2 b_4^3 q^{-3} [3]_1^{-1} + b_3 b_5 q^{-3}$ ,  $b_6 b_3 = b_3 b_6 - b_4^2 \langle 1 \rangle q^{-1}$ ,  $b_5 b_4 = b_4 b_5 q^{-3}$ ,  $b_6 b_4 = [3]_1 b_5 + b_4 b_6 q^{-1}$ ,  $b_6 b_5 = b_5 b_6 q^{-3}$ .

**Lemma 16.** For  $\tilde{E}_2^{a,b,c,d,e,f} = b_1^a b_2^b \dots b_6^f$ , we have

$$\begin{aligned} \tilde{E}_2^{a,b,c,d,e,f} \cdot e_1 &= \tilde{E}_2^{a,b,c,d,e,f+1}, \\ \tilde{E}_2^{a,b,c,d,e,f} \cdot e_2 &= -\langle 1 \rangle [e]_2 q^{-3c-d+3f-1} \tilde{E}_2^{a,b+1,c,d+1,e-1,f} \\ &\quad + \langle 1 \rangle^2 [e-1]_2 [e]_2 [3]_1^{-1} q^{-3e+3f+3} \tilde{E}_2^{a,b,c,d+3,e-2,f} \\ &\quad - \langle 3 \rangle [d-1]_1 [d]_1 q^{-3c-2d+3e+3f+1} \tilde{E}_2^{a,b+1,c+1,d-2,e,f} \\ &\quad - \langle 1 \rangle [d]_1 q^{-6c-d+3(e+f)} \tilde{E}_2^{a,b+2,c,d-1,e,f} \\ &\quad + [f-1]_1 [f]_1 q^{-3e+f-2} \tilde{E}_2^{a,b,c,d+1,e,f-2} \\ &\quad + [3]_1 [d]_1 [f]_1 q^{2f-2d} \tilde{E}_2^{a,b,c+1,d-1,e,f-1} \\ &\quad + [f]_1 q^{-3c-d+2f-2} \tilde{E}_2^{a,b+1,c,d,e,f-1} \\ &\quad + q^{-3(b+c-e-f)} \tilde{E}_2^{a+1,b,c,d,e,f} \\ &\quad + \langle 1 \rangle^2 [c]_2 [3]_1^{-1} q^{3(-2c+e+f+1)} \tilde{E}_2^{a,b+3,c-1,d,e,f} \\ &\quad - \langle 3 \rangle [d-2]_1 [d-1]_1 [d]_1 q^{3(-d+e+f+2)} \tilde{E}_2^{a,b,c+2,d-3,e,f} \\ &\quad - \langle 1 \rangle [e]_2 [f]_1 q^{-3e+2f} \tilde{E}_2^{a,b,c,d+2,e-1,f-1} \\ &\quad - [e]_2 q^{-3d+3f} (q^{2d+1} [3]_1 - [2]_2) \tilde{E}_2^{a,b,c+1,d,e-1,f} \\ &\quad + [f-2]_1 [f-1]_1 [f]_1 \tilde{E}_2^{a,b,c,d,e+1,f-3}, \\ e_1 \cdot \tilde{E}_2^{a,b,c,d,e,f} &= -\langle 1 \rangle [c]_2 q^{3a+b-3c+2} \tilde{E}_2^{a,b,c-1,d+2,e,f} \end{aligned}$$



$$\begin{aligned}
& + [3]_1 [b-1]_1 [b]_1 q^{3a-b+2} \tilde{E}_2^{a,b-2,c+1,d,e,f} \\
& + [3]_1 [d]_1 q^{3a+b-2d+2} \tilde{E}_2^{a,b,c,d-1,e+1,f} \\
& + q^{3a+b-d-3e} \tilde{E}_2^{a,b,c,d,e,f+1} \\
& + [2]_1 [b]_1 q^{3(a-c)} \tilde{E}_2^{a,b-1,c,d+1,e,f} \\
& + [a]_2 \tilde{E}_2^{a-1,b+1,c,d,e,f}. \\
e_2 \cdot \tilde{E}_2^{a,b,c,d,e,f} &= \tilde{E}_2^{a+1,b,c,d,e,f}.
\end{aligned}$$

*Proof.* By induction, we have

$$\begin{aligned}
b_6 b_1^n &= q^{3n} b_1^n b_6 + [n]_2 b_1^{n-1} b_2, \\
b_6 b_2^n &= [3]_1 q^{2-n} [n-1]_1 [n]_1 b_2^{n-2} b_3 + q^n b_2^n b_6 + [2]_1 [n]_1 b_2^{n-1} b_4, \\
b_4 b_3^n &= q^{-3n} b_3^n b_4, \\
b_6 b_3^n &= b_3^n b_6 - \langle 1 \rangle q^{2-3n} [n]_2 b_3^{n-1} b_4 b_4, \\
b_6 b_4^n &= [3]_1 q^{2-2n} [n]_1 b_4^{n-1} b_5 + q^{-n} b_4^n b_6, \\
b_6 b_5^n &= q^{-3n} b_5^n b_6, \\
&\text{and} \\
b_6^n b_1 &= q^{n-2} [n-1]_1 [n]_1 b_4 b_6^{n-2} + q^{3n} b_1 b_6^n \\
&\quad + q^{2(n-1)} [n]_1 b_2 b_6^{n-1} + [n-2]_1 [n-1]_1 [n]_1 b_5 b_6^{n-3}, \\
b_5^n b_1 &= \langle 1 \rangle^2 q^{-3(n-1)} [n-1]_2 [n]_2 [3]_1^{-1} b_4^3 b_5^{n-2} + q^{3n} b_1 b_5^n \\
&\quad - q^{-3} (q^4 + q^2 - 1) [n]_2 b_3 b_5^{n-1} - q^{-1} \langle 1 \rangle [n]_2 b_2 b_4 b_5^{n-1}, \\
b_5^n b_2 &= b_2 b_5^n - \langle 1 \rangle q^{2-3n} [n]_2 b_4 b_4 b_5^{n-1}, \\
b_5^n b_4 &= q^{-3n} b_4 b_5^n, \\
b_4^n b_1 &= -\langle 3 \rangle q^{6-3n} [n-2]_1 [n-1]_1 [n]_1 b_3^2 b_4^{n-3} - \langle 1 \rangle q^{-n} [n]_1 b_2^2 b_4^{n-1} \\
&\quad - \langle 3 \rangle q^{1-2n} [n-1]_1 [n]_1 b_2 b_3 b_4^{n-2} + b_1 b_4^n, \\
b_4^n b_2 &= [3]_1 q^{2-2n} [n]_1 b_3 b_4^{n-1} + q^{-n} b_2 b_4^n, \\
b_4^n b_3 &= q^{-3n} b_3 b_4^n, \\
b_3^n b_1 &= q^{-3n} b_1 b_3^n + \langle 1 \rangle^2 q^{3-6n} [n]_2 [3]_1^{-1} b_2^3 b_3^{n-1}, \\
b_3^n b_2 &= q^{-3n} b_2 b_3^n, \\
b_2^n b_1 &= q^{-3n} b_1 b_2^n.
\end{aligned}$$

The lemma is a direct consequence of these formulae.  $\square$

A part of the above results have also been obtained in [24].

Let us turn to the representations  $\pi_i$  of  $A_q(G_2)$ . The elements  $\sigma_i$  in Definition 3 and  $\sigma_i e_i$  are given by

$$\sigma_1 = t_{17}, \quad \sigma_2 = t_{26} t_{17} - q t_{27} t_{16}, \quad \sigma_1 e_1 = t_{27}, \quad \sigma_2 e_2 = t_{36} t_{17} - q t_{37} t_{16}. \quad (42)$$

From (14) and (27) with  $\alpha_i = \mu_i = 1$ , we have

$$\begin{aligned}
\pi_1(\sigma_1) &= \mathbf{k}_1 \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4 \mathbf{k}_5, \\
\pi_1(\sigma_2) &= \mathbf{K}_2 \mathbf{k}_3^3 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6, \\
\pi_1(\sigma_1 e_1) &= \mathbf{a}_1^+ \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4 \mathbf{k}_5, \\
\pi_1(\sigma_2 e_2) &= \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{k}_3^3 \mathbf{K}_4 \mathbf{A}_6^+ + [2]_2 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{K}_2 \mathbf{A}_4^+ \mathbf{K}_4 \mathbf{k}_5^3 \mathbf{K}_6 + \mathbf{a}_1^{-3} \mathbf{A}_2^+ \mathbf{k}_3^3 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 \\
&\quad + [3]_1 \mathbf{a}_1^{-2} \mathbf{k}_1 \mathbf{a}_3^+ \mathbf{k}_3^2 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4 \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6 - q[3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{A}_4^+ \mathbf{K}_4 \mathbf{k}_5^3 \mathbf{K}_6 \\
&\quad + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{a}_3^- \mathbf{k}_3^2 \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6 + \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{k}_3^3 \mathbf{A}_4^- \mathbf{a}_5^+ \mathbf{K}_6 + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{A}_2^- \mathbf{a}_3^+ \mathbf{k}_3 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 \\
&\quad + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{K}_2 \mathbf{a}_3^- \mathbf{k}_3 \mathbf{A}_4^+ \mathbf{K}_4 \mathbf{k}_5^3 \mathbf{K}_6 + \mathbf{k}_1^3 \mathbf{A}_2^{-2} \mathbf{a}_3^+ \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{K}_2 \mathbf{a}_3^+ \mathbf{k}_3 \mathbf{K}_4 \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6 \\
&\quad + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^{-3} \mathbf{A}_4^+ \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{a}_3^{-2} \mathbf{k}_3 \mathbf{A}_4^+ \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6, \\
\lambda_1^{-1} \pi_1(\xi_1) &= \mathbf{a}_1^+ \mathbf{K}_1^{-1}, \\
\lambda_2^{-1} \pi_1(\xi_2) &= \mathbf{a}_1^{-3} \mathbf{A}_2^+ \mathbf{K}_2^{-1} + [2]_2 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{k}_3^{-3} \mathbf{A}_4^+ \mathbf{K}_4^{-1} - q[3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{k}_3^{-1} \mathbf{A}_4^+ \mathbf{K}_4^{-1} \\
&\quad + [3]_1 \mathbf{a}_1^{-2} \mathbf{k}_1 \mathbf{K}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1} + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{a}_3^- \mathbf{k}_3^{-2} \mathbf{A}_4^+ \mathbf{K}_4^{-1} + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{k}_3^{-1} \mathbf{K}_4^{-1} \mathbf{a}_5^+ \mathbf{k}_5^{-1} \\
&\quad + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{K}_4^{-1} \mathbf{k}_5^{-3} \mathbf{A}_6^+ \mathbf{K}_6^{-1} + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{A}_2^- \mathbf{K}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-2} + [3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{a}_3^- \mathbf{k}_3^{-2} \mathbf{K}_4^{-1} \mathbf{a}_5^+ \mathbf{k}_5^{-1} \\
&\quad + \mathbf{k}_1^3 \mathbf{A}_2^{-2} \mathbf{K}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-3} + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^- \mathbf{k}_3^{-1} \mathbf{K}_4^{-2} \mathbf{a}_5^+ \mathbf{k}_5^{-2} + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{A}_4^- \mathbf{K}_4^{-2} \mathbf{a}_5^+ \mathbf{k}_5^{-3} \\
&\quad + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^{-3} \mathbf{k}_3^{-3} \mathbf{A}_4^+ \mathbf{K}_4^{-2} + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^{-2} \mathbf{k}_3^{-2} \mathbf{A}_4^+ \mathbf{K}_4^{-2} \mathbf{a}_5^+ \mathbf{k}_5^{-1}, \\
\pi_2(\sigma_1) &= \mathbf{k}_2 \mathbf{K}_3 \mathbf{k}_4^2 \mathbf{K}_5 \mathbf{k}_6, \\
\pi_2(\sigma_2) &= \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3^2 \mathbf{k}_4^2 \mathbf{K}_5, \\
\pi_2(\sigma_1 e_1) &= \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3 \mathbf{k}_4 \mathbf{a}_6^+ + \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4^2 \mathbf{K}_5 \mathbf{k}_6 + \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3 \mathbf{a}_4^- \mathbf{A}_5^+ \mathbf{k}_6 + \mathbf{K}_1 \mathbf{a}_2^{-2} \mathbf{A}_3^+ \mathbf{k}_4^2 \mathbf{K}_5 \mathbf{k}_6 \\
&\quad + [2]_1 \mathbf{K}_1 \mathbf{a}_2^- \mathbf{k}_2 \mathbf{a}_4^+ \mathbf{k}_4 \mathbf{K}_5 \mathbf{k}_6 + \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{A}_3^- \mathbf{a}_4^+ \mathbf{K}_5 \mathbf{k}_6, \\
\pi_2(\sigma_2 e_2) &= \mathbf{A}_1^+ \mathbf{k}_2^3 \mathbf{K}_3^2 \mathbf{k}_4^3 \mathbf{K}_5, \\
\lambda_1^{-1} \pi_2(\xi_1) &= \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + [2]_1 \mathbf{K}_1 \mathbf{a}_2^- \mathbf{K}_3^{-1} \mathbf{a}_4^+ \mathbf{k}_4^{-1} + \mathbf{K}_1 \mathbf{a}_2^{-2} \mathbf{k}_2^{-1} \mathbf{A}_3^+ \mathbf{K}_3^{-1} + \mathbf{K}_1 \mathbf{k}_2 \mathbf{a}_4^- \mathbf{k}_4^{-2} \mathbf{A}_5^+ \mathbf{K}_5^{-1} \\
&\quad + \mathbf{K}_1 \mathbf{k}_2 \mathbf{k}_4^{-1} \mathbf{K}_5^{-1} \mathbf{a}_6^+ \mathbf{k}_6^{-1} + \mathbf{K}_1 \mathbf{k}_2 \mathbf{A}_3^- \mathbf{K}_3^{-1} \mathbf{a}_4^+ \mathbf{k}_4^{-2}, \\
\lambda_2^{-1} \pi_2(\xi_2) &= \mathbf{A}_1^+ \mathbf{K}_1^{-1}.
\end{aligned}$$

We find that  $\pi_i(\sigma_i)$  is invertible. Comparing these formulae with Lemma 16 by using (40), the equality (37) is directly checked. Thus Proposition 12 is established for  $G_2$ .

In terms of the checked intertwiner  $\mathcal{F}$  in Section 4.4, Theorem 11 implies

$$E_2^{a,b,c,d,e,f} = \sum_{i,j,k,l,m,n} \mathcal{F}_{ijklmn}^{abcdef} E_1^{n,m,l,k,j,i}.$$

## 6. DISCUSSION

In view of Proposition 12 it is natural to expect that the map defined on generators of  $U_q^+(\mathfrak{g})$  as  $e_i \mapsto \eta_i := \sigma_i e_i / \sigma_i$  extends to an algebra homomorphism from  $U_q^+(\mathfrak{g})$  to  $A_q(\mathfrak{g})_{\mathcal{S}}$ , namely,  $\eta_i$  satisfies  $q$ -Serre relations. In fact, it is true not only for rank 2 cases but also for any  $\mathfrak{g}$ .

**Theorem 17.** *In  $A_q(\mathfrak{g})_{\mathcal{S}}$  the following relation holds for any  $i, j$  ( $i \neq j$ ):*

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \eta_i^{(r)} \eta_j \eta_i^{(1-a_{ij}-r)} = 0.$$

*Proof.* By relabeling of Dynkin indices we can assume  $i = 1, j = 2$ . Set  $\tau_i = \sigma_i e_i$  for  $i = 1, 2$ . Then from Proposition 4 we have

$$\sigma_i \tau_i = q_i \tau_i \sigma_i \quad (i = 1, 2), \quad \sigma_i \tau_j = \tau_j \sigma_i \quad (i, j = 1, 2; i \neq j). \quad (43)$$

Using (12) with these relations one verifies

$$\eta_1^r \eta_2 \eta_1^s = q_1^{(r+s)(r+s-1)/2} (\tau_1^r \tau_2 \tau_1^s) / (\sigma_1^r \sigma_2 \sigma_1^s).$$

Here we have set  $s = 1 - a_{12} - r$ . Recalling that  $\sigma_1$  and  $\sigma_2$  commute with each other, we can reduce the claim to showing

$$Z := \sum_{r=0}^{1-a_{12}} (-1)^r \tau_1^{(r)} \tau_2 \tau_1^{(s)} = 0.$$

Note that the right (resp. left) weight of  $Z$  is  $(1 - a_{12})(\varpi_1 - \alpha_1) + (\varpi_2 - \alpha_2)$  (resp.  $w_0((1 - a_{12})\varpi_1 + \varpi_2)$ ). The two weights are not related by the longest element  $w_0 \in W$ . Hence if we show  $f_i Z = Z f_i = 0$  for any  $i$ , we can conclude  $Z = 0$  by the remark after Lemma 5. The properties  $f_i Z = 0$  for any  $i$  and  $Z f_i = 0$  for  $i \neq 1, 2$  are trivial.

First we show  $Z f_2 = 0$ . We have

$$(\tau_1^r \tau_2 \tau_1^s) f_2 = \tau_1^r (\tau_2 f_2) (\tau_1 k_2^{-1})^s = \tau_1^r \sigma_2 (\beta \tau_1)^s = \beta^s \tau_1^{r+s} \sigma_2,$$

where  $\beta = q_2^{-\langle h_2, \varpi_1 - \alpha_1 \rangle} = q_2^{a_{21}} = q_1^{a_{12}}$  and we have used (43). Hence,

$$Z f_2 = \left( \sum_{r+s=1-a_{12}} \frac{(-q_1^{-a_{12}})^s}{[r]_1! [s]_1!} \right) (-\tau_1)^{1-a_{12}} \sigma_2 = 0.$$

In the last equality we have used the following formula:

$$\sum_{i=0}^m (-z)^i \begin{bmatrix} m \\ i \end{bmatrix} = \prod_{j=1}^m (1 - z q^{2j-m-1}),$$

where  $\begin{bmatrix} m \\ i \end{bmatrix} = [m]! / ([i]! [m-i]!)$ .

Next, we show  $Z f_1 = 0$ .

$$\begin{aligned} (\tau_1^r \tau_2 \tau_1^s) f_1 &= \sum_{i=1}^r \tau_1^{r-i} \sigma_1 (\tau_1 k_1^{-1})^{i-1} (\tau_2 k_1^{-1}) (\tau_1 k_1^{-1})^s + \tau_1^r \tau_2 \sum_{i=1}^s \tau_1^{s-i} \sigma_1 (\tau_1 k_1^{-1})^{i-1} \\ &= \sum_{i=1}^r \delta \gamma^{i-1-s} \tau_1^{r-1} \tau_2 \tau_1^s \sigma_1 + \sum_{i=1}^s \gamma^{i-1} \tau_1^r \tau_2 \tau_1^{s-1} \sigma_1, \end{aligned}$$

where constants  $\gamma, \delta$  are determined by  $\sigma_1(\tau_1 k_1^{-1}) = \gamma \tau_1 \sigma_1$ ,  $\sigma_1(\tau_2 k_1^{-1}) = \delta \tau_2 \sigma_1$  and hence we have  $\gamma = q_1 q_1^{-\langle h_1, \varpi_1 - \alpha_1 \rangle} = q_1^2$ ,  $\delta = q_1^{-\langle h_1, \varpi_2 - \alpha_2 \rangle} = q_1^{a_{12}}$ . Then, we obtain

$$\begin{aligned} Z f_1 &= \sum_{r+s=1-a_{12}} \frac{(-1)^r}{[r]_1! [s]_1!} \left( \sum_{i=1}^r \delta \gamma^{i-1+s} \tau_1^{r-1} \tau_2 \tau_1^s \sigma_1 + \sum_{i=1}^s \gamma^{i-1} \tau_1^r \tau_2 \tau_1^{s-1} \sigma_1 \right) \\ &= \sum_{r+s=1-a_{12}} \frac{(-1)^r}{[r]_1! [s]_1!} \left( \delta \gamma^s \frac{1 - \gamma^r}{1 - \gamma} \tau_1^{r-1} \tau_2 \tau_1^s \sigma_1 + \frac{1 - \gamma^s}{1 - \gamma} \tau_1^r \tau_2 \tau_1^{s-1} \sigma_1 \right) \\ &= \sum_{r+s=1-a_{12}} \left( -(-1)^{r-1} q_1^s \tau_1^{(r-1)} \tau_2 \tau_1^{(s)} + (-1)^r q_1^{s-1} \tau_1^{(r)} \tau_2 \tau_1^{(s-1)} \right) \sigma_1 = 0 \end{aligned}$$

as desired.  $\square$

By Theorem 17 we could embed  $U_q^+(\mathfrak{g})$  into  $A_q(\mathfrak{g})_{\mathcal{S}}$  or  $A_q(\mathfrak{g})$ . It will be interesting to investigate it further in the light of the quantum cluster algebra which has been recognized as a fundamental structure in the quantized algebra of functions [7]. The representations via multiplication on PBW bases also play a fundamental role in the study of the positive principal series representations and modular double [9].

In this paper we have not discussed the analogue of the tetrahedron and 3D reflection equations for general  $\mathfrak{g}$ . However, from our proof of Theorem 11, we expect that the basic constituents are  $R$  and  $K$  only, and their compatibility condition is reduced to the rank 2 cases (2) and (3).

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